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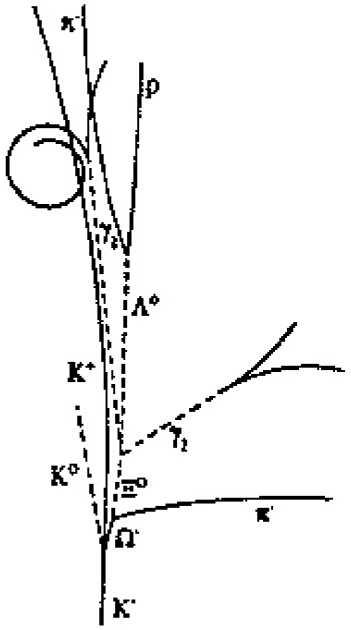

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QUANTUM DEGENERACIES IN BLACK HOLES AND THE AdS_5 SPACETIME

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Abstract

The recent derivation of an exact integer partition function (IPF) is related to a generating function for strings which compose the states of a black hole. This leads to Jacobi θ-function realizations. Within the context of AdS spacetimes and conformal completion this provides an E_6 gauge theory in the context of an AdS_5 spacetime.

Keywords: Stringy black holes, integer partition function, exceptional groups, Jordan matrix algebra, quantum encryption

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The Quantum Coded Universe

This paper addresses quantum gravity as quantum encryption system. The microstates of a black hole are modeled by the integer partition function. The modular structure, or more explicitly the Dedekind η-function, suggests the most general symmetry for quantum cosmology and gravity is the Jordan matrix algebra.

The simulation of quantum black holes within this structure will lead to quantum computing and encryption-communication technology of a general variety. A black hole may be optically simulated, where entangled qubits may be integrated into a quantum optical computer. The first step will be with the optical simulation of black holes with 4-partite entanglement configurations with 8 states. A triality condition, or $\mathbb{O}^{\otimes 3}$, of three sets of these will then be a simulation of a black hole which obeys the proper integer partition of microstates

Black holes as Degenerate States and the Partition function

The entropy of a black hole is a measure of the number of microstates, where for N degenerate microstates the entropy is $S = k\,log(N)$ associated with gravity [1]. Entropy for large N is determined by the area of the event horizon $S = kA/4L_p^2$, where for the Schwarzschild black hole $A = 16\pi M^2$. The black hole holds a set of states with energy $E = M$ in a degeneracy $g(E) = exp(4\pi E^2)$ and the partition function is [2]

$$Z(\beta) = \sum_E e^{4\pi E^2} e^{-\beta E}. \tag{1}$$

This partition function is divergent for $E \rightarrow \infty$. The statistics for the number of degenerate microstates for a black hole is unbounded, and the partition function diverges. Black hole entropy is a coarse graining microstate states, which has been accomplished in string theory for large N. The horizon area is a summation of these quantum numbers

$$A = 16\pi\alpha \sum_{i=1}^{N} n_i, \tag{2}$$

for α_p a Planck area. The quantum numbers n_i determine an element of the horizon area. The energy is then counted as $E_n = \alpha E_p \sqrt{n}$, for $n = \sum_{i=1}^{N} n_i$[2]

The degeneracy for E_n is the number of ways $n > 0$ is a sum of N or less positive integers n_i. It is the cardinality of the set of elements

$\{n_1, n_2, , \ldots, n_m\}$, such that $n = \sum_{i=1}^{m} n_i$ for $1 \leq m < N$. The number of ways a positive integer m may be written as a sum of m positive integers is the same problem as computing the number of ways of arranging n balls in m cells in a row. The result is a degeneracy for the energy E_n

$$g(E_n) = \sum_{m=1}^{N} \begin{pmatrix} n-1 \\ m-1 \end{pmatrix}, \tag{3}$$

for $N \leq n$. We also have that $m \leq n$, which cuts the degeneracy further in

$$g'(E_n) = \sum_{m=1}^{n} \begin{pmatrix} n-1 \\ m-1 \end{pmatrix}. \tag{4}$$

The partition function is a summation of the two degenerate sets,

$$Z(\beta) = \sum_{n=1}^{N}\sum_{m=1}^{n} \begin{pmatrix} n-1 \\ m-1 \end{pmatrix} e^{-E_p\alpha\sqrt{n}} + \sum_{n=M+1}^{\infty}\sum_{m=1}^{N} \begin{pmatrix} n-1 \\ m-1 \end{pmatrix} e^{-E_p\alpha\sqrt{n}}. \tag{5}$$

The two portions of the partition functions play a role at n small and $n >> N$, and may be computed independently [2]. The convergence occurs for $n >> N$ with

$$Z(\beta) \simeq \sum_{n=N+1}^{\infty} (n-1)^{N-1} e^{-\beta E_p\alpha\sqrt{n}} \tag{6}$$

This is a convergent partition function. Conversely for a low black hole temperature $n << N$, the degeneracy from the binomial theorem is $g'(E_n) \simeq 2^{n-1}$ and the black hole entropy is $S = k\, ln(2^{n-1}) = (n-1)ln2$. The area $A = 16\pi\alpha^2 n$ permits us to set $\alpha = \frac{1}{2}\frac{\sqrt{ln2}}{\pi}$. This gives the entropy of the black hole $S = A/4$.

The degeneracy is extended to consider the energy level as due to a spin with an elementary energy $E_j = \sqrt{j(j+1)}$. The spin is $SU(2)$, which is a double covering symmetry of the spherical gravity field for a Schwarzschild black hole, with $j = \frac{1}{2}$. There are two states $\{\frac{1}{2}, -\frac{1}{2}\}$ and for m spins there are 2^m possible spin configurations. Hence the degeneracy for energy n is

$$g(E_n) = \sum_{m=1}^{N} \begin{pmatrix} n-1 \\ m-1 \end{pmatrix} 2^m = 3^{n-1} \tag{7}$$

The entropy of the black hole is now $S = (n-1)ln\ 3$ and $\alpha = \frac{1}{2}\frac{\sqrt{ln\ 3}}{\pi}$. This is then in agreement with the result by Hod [3] that quasi-normal modes are

$$\omega_n = \alpha^2 = \frac{ln\ 3}{4\pi} + 2\pi i(n + \frac{1}{2}) + \ldots. \tag{8}$$

This is equivalent to a redefinition $\alpha^2 \rightarrow \alpha^2 + ln(j(j+1))/4\pi$. In appropriate units the normal modes for a black hole of mass M are

$$\omega_n = \frac{ln\ 3}{8\pi M} + \frac{i}{4M}(n + \frac{1}{2}) + \ldots \tag{9}$$

so the partition function of the black hole is

$$Z[\beta] = \sum_n e^{4\pi\omega_n} e^{-\beta\alpha\sqrt{n}}. \tag{10}$$

The string states in 26 dimensions is described by the Virasoro algebra. The generating function $Tr(q^N)$ is the density of string states for $q = e^{2\pi iz}$ and $N = \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n$ the string number operator. The trace $Tr(q^N)$ over 24 transverse string oscillations is

$$Tr(q^N) = \prod_{n=1}^{\infty} Tr(q^n) = \prod_{n=1}^{\infty}(1 - q^n)^{-24}, \tag{11}$$

where $\prod_{n=1}^{\infty}(1 - q^n)$ is the partition function[4]. q^n counts the number of way to write $n = a_1 + 2a_2 + \ldots\ ma_m\ \ldots$, for a_i a natural number. This partition function is related to the Dedekind η-function

$$\eta = q^{\tau/2}\prod_{n=1}^{\infty}(1 - q^{n\tau}) \tag{12}$$

for $Tr(\rho^N) = q^{24\tau}\eta^{-24}$. The generating function written as

$$Tr(q^N) = f(q) = exp\Big(\sum_{n=1}^{\infty} ln(1 - q^n)\Big) = \sum_{n=0}^{\infty} p(n)q^n \tag{13}$$

which is the $p(n)$ partition of the integers This function $p(n)$ is approximated with the Hardy-Ramanujan approximate formula for the partition of the integers. The quantum statistics of states on a black hole are similarly

computed, and such generating functions should be modular. These may be Ramanujan mock Θ functions, as suggested by Dyson [5].

In both the case of black holes and the string generating function, the fundamental issue involves the partitioning of states as partitions of integers. The last equation for $Tr(q^N)$ contains the complex term $e^{2\pi inz}$, for $nz = 4\omega_n - ln\ 3/4\pi$. The remainder is an approximation for the integer partition function

$$p(n) \simeq 3e^{-\beta\alpha\sqrt{n}} \tag{14}$$

By using the Hardy-Ramanujan approximation for the integer partition the result is

$$p(n) \simeq \frac{1}{4n\sqrt{3}} e^{4\pi\sqrt{n}}, \tag{15}$$

which is remarkably similar formula to that obtained for the black hole.

This suggests the solution to the partition function is a partitioning of the integers, and the statistics are accounted for by an IPF. Recently a partition function has been derived as the finite sum and an algebraic number. This partition function $p(n)$ computes the partitions of an integer n, which is the sequence of integers:

1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490,

$$627,\ 792,\ 1002,\ \ldots,\ p(100) = 190569292,\ \ldots \tag{16}$$

The general form for $p(n)$ has been a major problem in number theory. The circle method of Hardy and Ramanujan produced an asymptotic approximation function

$$p(n) \simeq \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2n/3}} \tag{17}$$

Bruinier, Folsom, Kent and Ono [6][7] derived a formula for $p(n)$ as a finite sum of algebraic numbers using the Dedekind η function $\eta = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ and the Eisenstein series

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sum_{d|n} dp^n \tag{18}$$

A merimorphic form $F(z)$ with weight -2 by the discrete group $\Gamma_0(6)$ is defined according to Eisenstein series

$$F(z) = \frac{1}{2}\frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 5E_2(6z)}{(\eta(z)\eta(2z)\eta(3z)\eta(6z))^2}$$

$$= q^{-1} - 10 - 29q - 104q^3 - 273q^3 - \ldots \tag{19}$$

which is eigenvalued on the upper half of the complex plane $\Delta E_k(q) = k(k-1)E_k(q)$, and is a Maass form.

$$P(z) = -\Big(\frac{1}{2\pi i}\frac{d}{dz} + \frac{1}{2\pi y}\Big)F(z) \tag{20}$$

that has an eigenvalue -2 with respect to the hyperbolic Laplacian $\Delta = -y^2(\partial_x^2 + \partial_y^2)$.

The group $\Gamma_0(6)$ acts on these modular forms to construct a $\Gamma_0(6)$ equivalence classes $\mathcal{Q}_n$. The anti-holomorphic form $P(z)$ on this class at discrete point in the hyperbolic half-plane are summed over in a trace formula

$$tr(n) = \sum_{Q\in\mathcal{Q}_n} P(z_Q) \tag{21}$$

where $x_Q \in Q(x,y)$. This then gives a formula for the integer partition $p(n) = \frac{1}{2n+1}Tr(n)$

The integer partition function is constructed in the same manner partition functions are derived in statistical mechanics. The trace formula for the string states is a form of statistical mechanical device, and this mathematics gives a precise way in which these states may be partitioned to form an over all Boltzmann-like distribution, or e^{-iS} in a path integral.

Quantum Gravity as $\mathbb{O}^{\otimes 3}$

The application of the partition of integers to quantum states of gravity means they have representations as modular functions. The exceptional groups have Jacobi θ-function representations. The largest of the exceptional groups is the E_8 group, which does not have a complex representation, but subgroups do, such as $E_6 \times SU(3)$. Further, this may be overcome by extending the heterotic group into the Jordan matrix algebra.

The Jordan algebra is a 3×3 matrix comprised of three scalars and $\mathbb{O}_i$, for $i = 1, 2, 3$. The 16 supersymmetric partners are the two defined in two sets of spinor fields which are the superpartner to 8 vector fields which define three octionionic valued fields or E_8's. These exist in a Jordan exceptional algebra [8] of the 3×3 matrix with diagonal scalar elements and octonionic

diagonal elements

$$J^3(\mathbb{O}) = \begin{pmatrix} z_1 & \mathbb{O}_1 & \mathbb{O}_2^* \\ \mathbb{O}_1^* & z_2 & \mathbb{O}_3 \\ \mathbb{O}_2 & \mathbb{O}_3^* & z_3 \end{pmatrix}, \tag{22}$$

which has $27 = 24 + 3$ variables or dimensions. The F_4 group is the set of automorphisms of the $J^3(\mathbb{O})$, with the three polynomials in these 27 dimensions, These polynomials are the result of the trace of the Jordan matrix, the Jordan product $J^2 = J \circ J$, for $A \circ B = \frac{1}{2}(AB + BA)$, the Freudenthal product

$$J * J = J^2 - tr(J)J + tr(J \cdot J)I, \ \sigma(J) = tr(J \cdot J) \tag{23}$$

and the determinant.$det(J) = \frac{1}{3})tr(J \cdot J)oJ)$. The polynomial invariants are

$$Tr(J) = x + y + z$$

$$\sigma(J) = xy + yz + xz - \mathbb{O}_1\mathbb{O}_1^* - \mathbb{O}_2\mathbb{O}_2^* - \mathbb{O}_3\mathbb{O}_3^* \tag{24}$$

$$det(J) = xyz + (\mathbb{O}_1\mathbb{O}_3)\mathbb{O}_2 + \mathbb{O}_2^*(\mathbb{O}_1^*\mathbb{O}_3^*) - z|\mathbb{O}_1|^2 - y|\mathbb{O}_2|^2 - x|\mathbb{O}_3|^2.$$

Consider a theory with the Jacobi matrix algebra, with its θ-function realization. The 3 scalars of the 27 dimensional $J^3(\mathbb{O})$ on a light cone reduce to a two dimensional space $\sim \mathbb{R}^2$ and the three octonions form ($\mathbb{O}_v$, $\mathbb{O}_s$, $\mathbb{O}_{s'}$ are vector plus spinor elements, related to each other in $N = 8$ supersymmetry, where the two spinor elements are conjugates. The relevant elements are the 24 elements of the Jordan matrix which form the transverse oscillators. The Leech lattice θ-function series is $\Theta_{\Lambda_{24}}(z) = \Theta_{E_8}(z)^3 - 720\Delta_{24}(z)$, where $\Delta_{24}(z) = \sum_{n=0}^{\infty}\tau(n)q^n$ and $\tau(n)$ are Ramanujan numbers [9]. $\Delta_{24} = q\prod_n(1 - q^n)^{24}$, for $q = e^{2\pi iz}$, which gives the Dedekind η-function $\eta(q) = (\Delta_{24})^{1/24}$.

We return to the discrete structure, in particular with the conformal completion of the AdS, which ties in with the conformal flow onto the Einstein field equation. F_4 as the automorphism group defines the invariants of the Jordan algebra. This is in particular with the structure induced on the conformal completion of the AdS and the renormalization group flow. This structure is the discrete structure on light cones and its corresponding Heisenberg group. This structure is inherited through the Jacobi θ-functions

of the E_8 and $J^3(\mathbb{O})$ groups. The E_8 lattice, as well as the 24 dimensional Leech lattice Λ_{24} in the $26-27$ dimensional $J^3(\mathbb{O})$ and Lorentzian spacetime, have associated θ-functions

$$\Theta_\lambda(\tau) \;=\; \sum_{x\in\lambda} e^{i\pi\tau|x|^2}.\; Im\tau > 0. \tag{25}$$

The theta function of a lattice is then a holomorphic function on the upper half of the argand plane. The θ-function is an even unimodular lattice of rank n is actually a modular form of weight $n/2$. The E_8 lattice is proportional to a modular form of weight 4, the Eisenstein series $G_4(\tau)$. For $q \;=\; e^{i\pi\tau}$ the θ-function for the E_8 lattice is

$$\Theta_{E_8}(\tau) \;=\; 1 \;+\; 240\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}, \tag{26}$$

for $\sigma_3(n)$ a divisor function [9]. It is possible to expand this out in an explicit series, in particular if it is also written according to the Jacobi θ-functions:

$$\Theta_{E_8}(\tau) \;=\; \frac{1}{2}(\theta_2(q)^8 \;+\; \theta_3(q)^8 \;+\; \theta_4(q)^8), \tag{27}$$

for,

$$\theta_2(q) \;=\; \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}, \theta_3(q) \;=\; \sum_{n=-\infty}^{\infty} q^{n^2},$$

$$\theta_4(q) \;=\; \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}, \tag{28}$$

from which an explicit series may be computed.

The general Jacobi theta function

$$\theta(z,\;\tau) \;=\; \sum_{n=-\infty}^{\infty} exp(i\pi n^2\tau \;+\; 2\pi inz) \;=\; \sum_{n=-\infty}^{\infty} q^{n^2}\xi^n \tag{29}$$

for $q \;=\; exp(i\pi\tau)$ and $\xi \;=\; exp(2\pi iz)$. This is a modular form, or Jacobi form, that obeys the identities [10]

$$\theta(z \;+\; 1,\;\tau) \;=\; \theta(z,t),$$

$$\theta(z \;+\; a \;+\; b\tau,\;\tau) \;=\; exp(-\pi ib^2\tau \;-\; 2\pi ibz)\theta(z,\tau), \tag{30}$$

for a and b integers. The second of these is related to a particularly important property of Jacobi matrices. This leads to the fact that there is a Heisenberg group for this system. This is a θ-function representation of the Heisenberg group. $H_3(R)$. Consider a holomorphic function on variables defined on the upper half-plane, or $\tau > 0$. We then consider the translation operators on the function $f(z)$ so

$$(T_a f)(z) = f(z + a)$$

$$(U_b f)(z) = exp(\pi i b^2 \tau + 2\pi i b z) f(z + b\tau), \tag{31}$$

here the second of these emulates the symmetries of the θ-function. These two elements obey one parameter group properties $T_a \circ T'_a = T_{a+a'}$ and $U_b \circ U'_b = U_{b+b'}$. However the composite of the two elements differs with

$$(U_b T_a f)(z) = (U_b f)(z + a) = exp(\pi i b^2 \tau + 2\pi i b(z + a)) f(z + a + b\tau)$$

$$(T_a U_b f)(z) = exp(\pi i b^2 \tau + 2\pi i b z)(T_a f)(z + b\tau) \tag{32}$$

$$= exp(\pi i b^2 \tau + 2\pi i b z) f(z + a + b\tau),$$

which are not equal and in particular

$$(U_b T_a f)(z) - (T_a U_b f)(z) = [exp(2\pi i b a) - 1] exp(\pi i b^2 \tau + 2\pi i b z) f(z + a + b\tau)$$

$$= [exp(2\pi i b a) - 1](T_a U_b f)(z) \tag{33}$$

$$(U_b T_a f)(z) = exp(2\pi i b a)(T_a U_b f)(z).$$

The two operator products are related to each other by a unitary phase, which is a $U(1)$ group and there are two shift operations which correspond to two copies of the reals. So a general group element is

$$U_t(e^{i\phi}, a, b) = e^{i\phi}(T_a U_b f)(z) = e^{i\phi} exp(\pi i b^2 \tau + 2\pi i b z) f(z + a + b\tau). \tag{34}$$

The equationU(1)equation element is a center of this group, the commutative subgroup. If the phase angle is zero it is straight forwards to show this is a

nilpotent group of 3×3 matrices, which is the Heisenberg group above with the identification

$$M(a,b,\phi) \;=\; \begin{pmatrix} 1 & a & \phi \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \tag{35}$$

which is identified with their respective operators. The Heisenberg group is a θ-function representation isomorphic to the Weyl representation. The representation of the matrix element M has a representation by the action of R on $L^2(R)$ such that

$$R(M(a,b,\phi))\psi(x) \;=\; exp(ibx \;+\; i\phi)\psi(x \;+\; a),\; \psi \;\in\; L^2(R). \tag{36}$$

An explicit θ-function can be derived with a discrete subgroup of the Heisenberg algebra and we set $a,\; b \;\in\; Z$. A wave function formed from an infinite Fourier sum of these terms is then just the Jacobi θ-function, which may be easily verified. Further, the properties of the Heisenberg group produce the symmetry properties of the Jacobi θ-function.

The Jacobi θ function $\theta(x,t)$, for x real, obeys the Schrodinger equation

$$i\frac{\partial}{\partial t}\theta(x,\; t) \;=\; \frac{1}{4\pi}\frac{\partial^2}{\partial x^2}\theta(x,\; t), \tag{37}$$

or what is more commonly called a heat equation for $t \;\rightarrow\; it$. The construction clearly has quantum mechanical aspects, which referring back to the partition function, or degeneracy, for strings and black holes illustrates quantum foundations. The Heisenberg group is seen later to be related to the equivalency of the $SLOCC$ group of n-partite entanglements and spacetime moduli [11].

E_6, Anti-de Sitter spaces and Parabolic groups

The E_8 group may be decomposed into $E_6 \times SU(3)$. The subgroup E_6 has a complex representation, unlike a single E_8, which makes it a reasonable gauge group, and is a reasonable gadget to work with within SYM theories for a GUT, where one can get a 27-dimensional irreducible representation for the transformations of Dirac field and bosons.

The E_6 decomposes into maximal subalgebras which can be read off the Dynkin diagram, are the algebras of $SO(10) \times U(1)$ and $SU(6) \times SU(2)$.

The $SO(10)$ in spacetimes is $SO(9,\ 1)$, and the spinor representation of E_6 describes the Dirac equation for massless particles. The spacetime Dirac equation in spacetime is projected from 10 dimensions with $SL(2,\mathbb{O}) \rightarrow SL(2,\mathbb{C})$. The Dirac equation is $\delta\psi = 0$, where the projector δ is the $SL(2,\mathbb{O})$ 2×2 self adjoint octonionic matrix [8]. The state vector $\psi \in \mathbb{O}^{\otimes 2}$. Antisymmetry requires that $\delta \rightarrow P\delta - Tr(P)I$. The two component octonionic matrix means $\delta = \pm vv^\dagger$ and $\psi = v\phi$. The elements $v \in \mathbb{O}$ are quaternionic (Dirac matrix valued) and the state vector fermionic.

The octonions are pairs of quaternions, and so the general supersymmetric form of the octonion is

$$\Psi = \begin{pmatrix} \nu \\ \bar{\phi} \end{pmatrix},\ \Delta = \Psi\Psi^\dagger = \begin{pmatrix} \delta & \psi \\ \psi^\dagger & \bar{\phi}\phi \end{pmatrix}, \tag{38}$$

Such that $det(\Delta) = \bar{\phi}\phi det(\delta) + 2\delta\cdot vv^\dagger$, for $A\cdot B = \frac{1}{2}Tr(A\circ B) - Tr(A)Tr(B)$ and $A\circ B$ the Jordan product. The Lorentz properties of $SO(9,1)$ operator Δ with $M \in SL(2,\mathbb{O})$ and

$$\mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \tag{39}$$

The $SL(2,\mathbb{C})$ in ordinary spacetime with 3 boosts and 3 spatial rotations is extended to 21 $SO(7)$ rotations and 9 boosts. $det(\Delta)$ then contains a Lorentzian inner product in $10 = 9 + 1$ spacetime that is preserves the determinant in $\mathcal{M}\delta\mathcal{M}^\dagger$. Consequently we have $SL(2,\mathbb{O}) \subset E_6$, and is a double cover on $SO(9,1)$. This is the form of the Cayley spinor.

The quotient $E_6/B_4 = E_6/SO(9)$ contains $SO(9,1)/SO(9)$. Further, since E_6 decomposes into F_4 the Cayley plane constructed from F_4/B_4 is contained in E_6/B_4. Let the group $SO(9) \rightarrow SO(8,1)$, so the operator M is Lorentzian. The quotient group $SO(8,2)/SO(8,1) \simeq AdS_9$. Embedded in the construction of E_6/B_4 is the anti-de Sitter space in 9 dimensions. It is then clear that AdS spaces may be constructed in noncompact forms of E_6.

A noncompact form of E_6 is the determinant preserving transformation of $J^3(\mathbb{O})$. This is a real form with Killing form -26 and is denoted as $E_{6(-26)}$ [8], which has the inclusion map $F_4 \hookrightarrow E_{6(-26)}$, where F_4 is a maximal compact subgroup. The determinant preserving E_6 constructs the projective structure of $\mathbb{O}P^2$, and $E_{6(-26)}$ is a set of transformation which preserves lines. Hence $E_{6(-26)}$ is a group of collineations on $\mathbb{O}P^2$, which fixes a point in $SO(9,1)$. The isometries of $\mathbb{O}P^2$ are F_4, and so the inclusion map and

$SO(9) \rightarrow SO(9,1)$ induces $isom(\mathbb{O}P^2 \rightarrow coll(\mathbb{O}P^2)$ [8]. The noncompact real form of E_6 is the group of collineations of the octonionic projective plane OP2. It is also the group of determinant-preserving linear transformations of the exceptional Jordan algebra $J^3(\mathbb{O})$, in 27-dimensions. Consequently the compact real form of E_6 has a 27-dimensional complex representation.

There exists a 3-way cyclic permutation symmetry of $SO(9,1)$. The 3-way cyclic permutations amount to ways that a 2×2 matrix may be embeded into a 3×3 matrix. The elements of E_6 may be constructed from these three sets of $SO(9,1)$. This triality defines $SO(8)$ as the transverse degrees of freedom of $SO(9,1)$. $SO(8)$ also preserves an axis of $SO(9)$. The physical states of interest are the coset $E_6/SO(8)$. The projector operators form from elements may define a KP^n space, $K = R,\ C,\ H$, such that for $p \in J^n(k)$ this is idempotent $p^2 = p$ and $Tr(p) = r$, for rank $r = 1,\ 2$ for points and lines. The space of 4×4 matrices with $\phi \in J^4(C)\ p = \phi\phi^\dagger$ construct CP^3 as a twistor theory. The Calabi-Yau twistor manifold $CP^{3|4}$ [12] embed this into a 5×5 matrix as $J^5(C) \simeq T \oplus J^4(C) \oplus C^4$ with

$$\begin{pmatrix} \psi & \phi \\ \phi^\dagger & x \end{pmatrix} \tag{40}$$

for $x \in R$, $\psi \in J^4(C)$ and $\phi \in C^4$. The Hermitian valued matrix on $J^4(C)$ has a representation in $SL(4,\ C) \sim SL(8,R)$ with 16 elements. This is also the Cayley spinor.

The decomposition of $E_6 \rightarrow SO(10)\times U(1)$ is a likely candidate for a grand unification. $SO(10)$ gives rise to $SU(3)\times SU(2)\times U(1)$ for the standard model. The 78 representation [13] decomposes as

$$\mathbf{78} \rightarrow \mathbf{45}_0 \oplus \mathbf{16}_{-3} \oplus \bar{\mathbf{16}}_3 \oplus \mathbf{1}_0 \tag{41}$$

where the adjoint **450** is the $SO(10)$, the $\mathbf{1}_0$ the $U(1)$, and $\mathbf{16}_{-3}$ $\bar{\mathbf{16}}_3$ spinors, in the complex $(O\otimes C)P^2$. The Dirac equation $P\psi = 0$ on the spinor and its complex conjugate reduce this over all 32 dimensional space to 30 dimensions with the group $SO(6,C) = spin(6)$. The 5×5 $J^5(C)$ is then $\sim spin(5)$. The coset $spin(6)/spin(5)$ defines the complex valued AdS_5 if these groups assume the hyperbolic forms $SO(4,2,C)/SO(4,1,C)$.

This AdS_5 is a part of a moduli space formed from $E_{6(6)}/SO(8,C)$, which are the physical states of the theory. The fermions are represented by the $SO(8)$

$$\frac{E_{6(6)}}{spin(8)} \rightarrow \frac{SO(10)\otimes(\mathbf{16}\otimes\mathbf{16}\otimes\mathbf{1})}{spin(7,1)} \rightarrow$$

$$\frac{SO(10) \times spin(4,2)}{spin(4,1)} \sim SO(10) \times AdS_5. \tag{42}$$

This is then a reasonable type IIB string theory on $AdS_5 \times S^5$ [14] The graviton sector of $E_6/spin(8)$ is a singlet in both the fermionic $spin(8)$, with twistor space connection above, and the gauge bosons in E_6. The graviton emerges in this anti-de Sitter spacetime. The correspondence $AdS_5 \sim CFT_4$ is easily then seen. For 32 supersymmetic generators and $8\mathcal{N} = 32$ we have the gauge group $SU(\mathcal{N}) = SU(4)$ This is the same group as the isometry group of the AdS_4 $spin(6) \simeq SU(4)$. We then have the appropriate CFT_4 $\sim$ $SU(4)$ which exists on the boundary of the $\partial AdS_5 = E_c^4$, which is a conformally flat spacetiime.

The quotient of E_6 by the fermionic sector in $spin(8)$ leads to the anti-de Sitter space. This structure is contained in the Twistor construction, where the quotient effectively "mods out" null congruencies. This null condition is found with the $\delta\psi = 0$ for the massless fermion. The null condition is contain in a parabolic group P which is derived from the compactification of the AdS_n space. It also corresponds to the derivation of Heisenberg group affiliated with the conformal compactificatio nof AdS_n. A quotient of the E_6 with the parabolic group P also defines the Cayley projective plane OP^2.

AdS_n Compactification, Light Cones and Klein Group structure

We digress to the issues of compactification of AdS spaces. The AdS_{n+1} group of isometries $O(n,2)$ contains a Mobius subgroup, or modular transformations, so that this discrete group does not necessarily act effectively on AdS_{n+1}. This means that the discrete group Γ is not necessarily convergent on the boundary space M_n. Such a convergence means there exists a sequence $g_i \in \Gamma$ which admits a north-south dynamics of poles $p^{\pm}$ on a sphere, which in the hyperbolic case defines the past and future portions of a light cone [15]. The limit set of a discrete group is a closed Γ-invariant subset that defines a $\Lambda_\Gamma \subset M_n$ so the complement Ω_Γ acts properly on M_n. This Γ-invariant closed subset of $\Lambda_\Gamma \subset L_n$ is the space of lightlike geodesic in M_n. This has some interesting properties. The action of Γ on $\Omega_\Gamma \cup AdS_{n+1}$ is contained in M_n. The open set Λ_Γ is the maximal set that the Γ acts properly on $\Omega_\Gamma \cup AdS_{n+1}$. The other is the discrete group Γ is Zariski dense in $O(n,\ 2)$.

The lightlike geodesics in M_n are copies of RP^1, which at a given point p define a set that is the lightcone $C(p)$ [15]. The point p is the projective action of $\pi(v)$ for v a vector in a local patch $R^{n,2}$ and so $C(p)$ is then $\pi(P \cap C^{n,2})$, for P normal to v, and $C^{n,2}$ the region on $R^{n,2}$ where the interval vanishes. The space of lightlike geodesics is a set of invariants and then due to a stabilizer on $O(n,2)$, so the space of lightlike curves L_n is identified with the quotient $O(n,2)/P$, where P is a subgroup defined the quotient between a subgroup with a Zariski topology, or a Borel subgroup, and the main group $G = O(n, 2)$. This quotient G/P is a projective algebraic variety, or flag manifold and P is a parabolic subgroup. The natural embedding of a group $H \rightarrow G$ composed with the projective variety $G \rightarrow G/P$ is an isomorphism between the H and G/P. This is then a semi-direct product $G = P \rtimes H$. For the G any $GL(n)$ the parabolic group is a subgroup of upper triangular matrices.

The G/P coset construction extends to the exceptional group E_6/P as the Cayley projective plane OP^2. This is a form of the holographic principle. The cyclic group for the complex form of E_6 is $Z/3Z$, which plays the role of the discrete group Γ. The conformal completion of AdS_5 is then determined by this discrete group, which is the fundamental group for the representation of 27 complex dimensional lines. The conformal completion of the AdS_5 is identified with the existence of light cones, null or projective structures, and the Heisenberg group. The physical motivating reason is the integer accounting of microstates and the degeneracy of a black hole. These require the use of modular functions, in particular those for E_8 and the Jordan matrix algebra. The gauge theory which most naturally emerges from this is E_6, where the physical states $E_6/spin(8)$ describe a gauge theory on AdS_5.

Physical Discussion

This indicates gravitation and quantum mechanics have a common origin. The conformal compactification of AdS_5 determines the existence of a CFT on a conformally flat spacetime. The double coset structure, or Kleinian system constructs the parabolic group or Borel subgroup structure for both the Heisenberg group and lightcone structure. The Heisenberg group also emerges as an aspect of the η-function realization for the degeneracy of states for a black hole. The moduli space of physical states in the $E_6/spin(8)$ probides an alternate route with the AdS_5 and the Heisenberg group from the Borel subgroup on the compactification of AdS_n

The boundary of the AdS_5 $\partial AdS_5 = E_c^4$ is a conformally flat spacetime [14]. The conformal transformation $g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$ the flat spacetime element $ds^2 = du^2 - \sum_i dx^i dx^i$ is a time dependent transformation for $du/dt = \Omega^{-1}$ and a de Sitter spacetime for $\Omega^2 = e^{\sqrt{\Lambda/3}\, t}$. This is approximately the spacetime for our physical universe. The gauge field on this spacetime which is equivalent to the the AdS_5 graviton theory is $SU(4)$. This is decomposed into $SU(3) \times U(1) \subset SU(4)$, where the $SU(3)$ theory is a form of QCD. The $SU(3)$ decomposed from the $SO(9,1)$ or $SO(10)$ is the color dual in an S-duality theory similar to the Seiberg duality between two supersymmetric QCDs. The gluon chain in the QCD affiliated with the AdS_5 boundary is then equivalent to a graviton, and S-dual supersymmetric transformation.

The spacetime which consists of the AdS_5 boundary has lightlike curves L_n identified with the quotient $O(n,2)/P \subset E_6/P$, where the last is the projective Cayley plane, or contained in the heavenly sphere. This invariant structure is coincident with the Heisenberg group. These two aspects of local physics are the absolutes, the coset structure for the moduli space of lightlike rays, or equivalently horizons conditions on a spacetime, such as the extremal conditions on black holes, is equivalent to a Heisenberg group corresponding to this quotient. This is a form of the correspondence between entanglement $SLOCC$ groups and the moduli space for black holes and nilpotent groups [11].

Quantum wave functions carry with them configuration variables where entangled states share the same configuration variables. The structure of these entanglements is equivalent to the moduli space for spacetime structures, such as the STU theory of black holes. The particular configuration variable can be transformed arbitrarily, which means that time and space may be configured arbitrarily. Time may be advanced in any arbitrary way possible, but these transformations are all invariant with respect to the moduli of the spacetime, which is equivalent to the entanglement group. The entanglement group is the SLOCC group for the teleportation of qubits, and this requires the irreversible loss of a entanglement.

The proposal by George Ellis for a block universe with a dynamic occurrence of time [16], or this crystallizing of time in various present periods on a spatial sheet of \$3\$-dimensions invokes quantum measurement. This crystallizing of a present moment involves the reduction of states, such as with decoherence or measurement, and these objectifications of a quantum system defines an instantiation of a present moment. Quantum mechanics has no

contextual reality in that the basis of vectors is not determined by anything in quantum mechanics [17]. It is selected for by the experimentalist, and the contextuality of quantum mechanics is found in the nonlocality of superposition and entanglements, and in the incommensurability of observables. Hence the contextual aspects of a quantum measurement are what quantum interpretations are centered upon. In the Copenhagen interpretation (CI) the cut is the classical-quantum dichotomy. Of course for this to be "absolute" you need an apparatus which is absolutely non-quantal, meaning its mass must be infinite, and you need an infinite number of experiments. That is fictional of course. The many worlds interpretation (MWI) indicates there is a splitting off of world according to eigenbases selected. The idea is then that the world continues to be quantum mechanical. However, that is still funny, for the world is split off according to the contextuality of the wave function decoherence, or equivalently by the eigen-basis chosen by the experimenter. There is then no noncontextual reality within quantum mechanics which sets the basis for decoherent outcomes, or a measurement. So this too is not a complete picture.

Quantum interpretations attempt to reduce the mysterious nature of quantum mechanics to our classical understanding, which has an intuitive sensory aspect to how we perceive the world. Yet at the end if the universe is entirely quantum mechanical it is unlikely that any of these schemes can ever work completely. The hidden problem is that QM has an inherently noncontextual property without reality, but how we interpret QM is contextual. This is a subtle contradiction.

The dynamic block universe seems reasonable to say the progression of time involves decoherent events, or what has been called wave function collapse. Yet in a subtle way this model has the above contradiction, just as does MWI or CI. This suggests a number of things. The appropriate question along these lines is with the equivalency between quantum entanglements and spacetime configurations. It might be that what constitutes a present time and issues of a "flow of time" may simply not be appropriate questions to ask.

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CONVARIANCE OF THE ALGEBRA OF CONSTRAINTS OF 4-D GENERAL RELATIVITY

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Abstract

We argue that the standard canonical treatment of GR breaks manifest spacetime covariance. We present new variables which carry a reducible representation of gauge transformations and spacetime diffeomorphisms. A proposal is presented for an action designed to realize these symmetries at the canonical level.

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1 Introduction

In the canonical formulation of general relativity (GR) one performs a 3+1 splitting into spatial variables which evolve in time. This seems to go directly against the principle of general covariance, where space and time manifestly appear on an equal footing. In this paper we will show for certain fomulations of general relativity that this covariance becomes manifestly broken at the canonical level. Our aim in this paper will be to establish that a certain field provides an off-shell realization of the algebra of spacetime diffeomorphisms, and we will propose an action for gravity based on this field, with a view toward restoration of the original covariance of the theory. The full test of such a restoration at the canonical level will be reserved for future research.

Consider a general transformation of coordinates $x \in M$, where M is a 4-dimensional spacetime manifold

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x). \tag{1}$$

The transformations (1) induce the following Lie bracket between vector fields $\xi, \zeta \in M$

$$\left[\xi^\mu \partial_\mu, \zeta^\nu \partial_\nu\right] = \left(\xi^\mu \partial_\mu \zeta^\nu - \zeta^\mu \partial_\mu \xi^\nu\right)\partial_\nu, \tag{2}$$

which we will for the purposes of this paper regard as a Lie algebra of general coordinate transformations.

We would like find a description of GR which realizes this algebra at the canonical level. For later comparison let us perform a decomposition of (2) into spatial and temporal parts. For the commutator between two spatial diffeomorphisms we should choose $\xi^\mu = \delta^\mu_i M^i$ and $\zeta^\mu = \delta^\mu_j M^j$. The corresponding algebra is given by

$$\left[N^i \partial_i, N^j \partial_j\right] = \left(M^i \partial_i N^j - N^i \partial_i M^j\right)\partial_j. \tag{3}$$

The commutator of two spatial diffeomorphisms is a spatial diffeomorphism, which shows that at the level of the vector fields, spatial diffeomorphisms should form a subalgebra of (2). Moving on to the commutator of a spatial with a temporal diffeomorphism, we make the identifications $\xi^\mu = \delta^\mu_i N^i$ and $\xi^\mu = \delta^\mu_0 N$, yielding

$$\left[N^i \partial_i, N \partial_0\right] = (N^i \partial_i N)\partial_0 - (N \dot{N}^i)\partial_i. \tag{4}$$

The commutator of a spatial with a temporal diffeomorphism is a linear combination of the two transformations.

Moving on to the commutator of two temporal diffeomorphisms, we choose $\xi^\mu = \delta^\mu_0 M$ and $\zeta^\mu = \delta^\mu_0 N$, which yields

$$[M\partial_0, N\partial_0] = (M\dot{N} - N\dot{M})\partial_0. \tag{5}$$

The commutator of two temporal diffeomorphisms is another temporal diffeomorphism, constituting a subalgebra of (2). This suggests for a theory invariant under general coordinate transformations, that the fully covariant algebra of spacetime diffeomorphisms should decompose into two subalgebras, a spatial one and a temporal one.

The Poisson algebra of hypersurface deformations has been computed by Teitelboim [2]

$$\begin{aligned} \{\vec{H}[\vec{N}], \vec{H}[\vec{M}]\} &= H_k[N^i\partial^k M_i - M^i\partial^k N_i]; \\ \{H[N], \vec{H}[\vec{N}]\} &= H[N^i\partial_i N] \\ [H[N], H[M]] &= H_i[(N\partial_j M - M\partial_j N)q^{ij}], \end{aligned} \tag{6}$$

where $H^\mu = (H, H_i)$ refer to the Hamiltonian and diffeomorphism constraints, smeared by a lapse function and shift vector $N^\mu = (N, N^i)$. If H and H_i generate spatial and temporal diffeomorphisms, then one would like to be able to make the identifications

$$H \longrightarrow \partial_0; \quad H_i \longrightarrow \partial_i. \tag{7}$$

Comparison of (6) with (3), (4) and (5) shows some similarities and some differences. The part of (6) involving purely spatial diffeomorphisms is isomorphic with (3). However, the second equation of (6) is isomorphic with (4) only when $\dot{N}^i = 0$, which means that the shift vector N^i must be independent of time. Additionally, aside from the issue of phase space structure functions q^{ij}, the third line of (6) is not isomorphic with (5). Since two Hamiltonian constraints Poisson-commute into a diffeomorphism (and not a Hamiltonian) constraint, then the interpretation is that the algebra (6) breaks manifest 4-dimensional diffeomorphism covariance.

There are various nonmetric formulations of GR where the basic variables include a gauge connection A^a_μ (See e.g. [3], [4]). In this paper we will focus on the Ashtekar formulation [4], demonstrating that even in this case there are similar discrepancies with (2) regarding the temporal parts of the Poisson algebra of constraints. With a view toward restoring this algebra to be more in line with (2), we will propose a new action for GR, called the instanton representation I_{Inst}. This paper will show that a certain subset of the variables of I_{Inst} forms an off-shell representation of the algebra (2), reserving the check for its full canonical realization for a separate paper.

The organization of this paper is as follows. Section 2 will establish the symmetry group of a 4-D gauge connection A^a_μ as $SO(3,C) * Diff.$ Our notation signifies that this symmetry is an off-shell symmetry, derived

independently of any equations of motion, Poisson brackets or canonical structure. Section 3 brings in the Ashtekar formulation which uses A^a_μ as one of the dynamical variables. We show that the original $SO(3,C)*Diff$ symmetry of A^a_μ is not realized at the canonical level. Section 4 proposes a new variable Ψ_{ae}, which in conjunction with A^a_μ forms the basis for our proposition. This proposition is provided in the discussion section in the form of an action $I_{Inst}[A,\Psi]$. We demonstrate that Ψ_{ae} preserves at least the spatial part of the $SO(3,C)*Diff$ algebra, relegating the temporal part for future research. The temporal part becomes of interest when one wishes to obtain a reduced phase space under $SO(3,C)*diff$, the spatial part of the fully covariant algebra.

On a final note regarding index conventions, symbols from the beginning of the Latin alphabet $a,b,c,\ldots$ will denote internal $SO(3,C)$ indices, while from the middle $i,j,k,\ldots$ will denote spatial indices corresponding to a 3-D spatial manifold Σ. Spacetime indices will be denoted by Greek symbols $\mu,\nu\ldots$, and in the canonical decomposition we will assume a spacetime of topology $M=\Sigma\times R$, where M is a globally hyperbolic manifold of 4-D spacetime.

2 Symmetry group of gauge connections

The form variation of a 4-dimensional $SO(3,C)$ gauge connection $A^a_\mu=(A^a_0,A^a_i)$ under gauge transformations and spacetime diffeomorphisms will be established in this section. We will show that A^a_μ forms an off-shell representation of the $SO(3,C)*Diff$ group. By this we mean that the algebra for this group closes at the covariant level independently of any equations of motion or symplectic structure. For notational purposes for this paper, we will distinguish the algebra $SO(3,C)*diff$ from $SO(3,C)*Diff$. In this notation the latter refers to full spacetime diffeomorphisms while the former refers to only the spatial parts of these diffeomorphisms, which excludes the temporal diffeomorphisms.

Under an infinitesimal $SO(3,C)$ gauge transformation $\delta_{\vec\eta}$, the connection A^a_μ transforms as [1]

$$\delta_{\vec\eta}A^a_\mu=-D_\mu\eta^a=-\partial_\mu\eta^a-f^{abc}A^b_\mu\eta^c, \tag{8}$$

which is the covariant derivative of the gauge parameter with $SO(3,C)$ structure constants f^{abc}. Under infinitesimal spacetime diffeomorphisms δ_ξ, the connection A^a_μ transforms according to the Lie derivative [1]

$$\delta_\xi A^a_\mu=\xi^\nu\partial_\nu A^a_\mu+(\partial_\mu\xi^\nu)A^a_\nu. \tag{9}$$

We will show that transformations (8) and (9) form a symmetry group of A^a_μ. More precisely, we will show without using any equations of motion or

Poisson brackets that the Lie bracket of these transformations acting on A^a_μ yields a linear combination of the same transformations acting on A^a_μ. Let us quote here the final results of the algebra which will be computed in this section. The final results of the $SO(3,C) * Diff$ algebra are

$$\left[\delta_{\vec{\theta}}, \delta_{\vec{\eta}}\right] A^a_\mu = -\delta_{\vec{\theta}\times\vec{\eta}} A^a_\mu; \quad \left[\delta_\xi, \delta_{\vec{\eta}}\right] A^a_\mu = -\delta_{(\delta_\xi,\vec{\eta})} A^a_\mu; \quad \left[\delta_\xi, \delta_\zeta\right] A^a_\mu = -\delta_{[\xi,\zeta]} A^a_\mu, \tag{10}$$

namely that $SO(3,C) * Diff$ is a symmetry group of a $SO(3,C)$ gauge connection A^a_μ. The remaining subsections will prove that this is indeed the case by explicit calculation.

2.1 The subalgebra of $SO(3,C)$ gauge transformations

First we will show that the algebra of $SO(3,C)$ gauge transformations closes on this field. Acting on (8) with a second gauge transformation $\delta_{\vec{\theta}'}$ and using (8), we have

$$\begin{aligned} \delta_{\vec{\theta}'}\delta_{\vec{\eta}} A^a_\mu &= -f^{abc}(\delta_{\vec{\theta}'} A^b_\mu)\eta^c = -f^{abc}(D_\mu \theta'^b)\eta^c \\ &= f^{abc}\eta^c \partial_\mu \theta'^b + \eta^c\left(\delta^{cf}\delta^{ag} - \delta^{cg}\delta^{af}\right) A^f_\mu \theta'^g. \end{aligned} \tag{11}$$

We must then compute the result with the transformations reversed

$$\begin{aligned} \delta_{\vec{\theta}'}\delta_{\vec{\eta}} A^a_\mu &= f^{abc}\eta^c \partial_\mu \theta'^b + A^f_\mu \eta^f \theta'^a - A^a_\mu(\vec{\eta}\cdot\vec{\theta}'); \\ \delta_{\vec{\eta}}\delta_{\vec{\theta}'} A^a_\mu &= f^{abc}\theta'^c \partial_\mu \eta^b + A^f_\mu \theta'^f \eta^a - A^a_\mu(\vec{\theta}'\cdot\vec{\eta}). \end{aligned} \tag{12}$$

Subtraction of the bottom line of (12) from the top line yields the commutator of the two transformations

$$\left[\delta_{\vec{\theta}'}, \delta_{\vec{\eta}}\right] = \partial_\mu(f^{abc}\theta'^b\eta^c) + f^{afc}A^f_\mu(f^{cge}\theta'^g\eta^e) = D_\mu(\vec{\theta}'\times\vec{\eta})^a, \tag{13}$$

which one recognizes as a $SO(3,C)$ gauge transformation with composite parameter $f^{abc}\theta'^b\eta^c$. Therefore the result is

$$\left[\delta_{\vec{\theta}'}, \delta_{\vec{\eta}}\right] A^a_\mu = -\delta_{\vec{\theta}'\times\vec{\eta}} A^a_\mu, \tag{14}$$

namely that the commutator of two $SO(3,C)$ gauge transformations acting on A^a_μ is a $SO(3,C)$ gauge transformation acting on A^a_μ. Therefore, the connection A^a_μ provides a realization of $SO(3,C)$ gauge transformations, seen as a symmetry group.

2.2 Mixed components of the algebra

Next we move to the commutator of a gauge transformation with a spacetime diffeomorphism. The variation by diffeomorphism of the variation by $SO(3,C)$ gauge transformation of A^a_μ is given by

$$\begin{aligned}\delta_{\vec{\xi}}(\delta_{\vec{\eta}}A^a_\mu) &= -\delta_\xi\big(\partial_\mu\eta^a + f^{abc}A^b_\mu\eta^c\big) = -f^{abc}\eta^c(\delta_\xi A^b_\mu)\\ &= -f^{abc}\eta^c\big(\xi^\nu\partial_\nu A^b_\mu + (\partial_\mu\xi^\nu)A^b_\nu\big),\end{aligned} \tag{15}$$

where we have used (8) and (9). Reversing the order of the transformations yields, after making use of (8) and (9),

$$\begin{aligned}\delta_{\vec{\eta}}(\delta_\xi A^a_\mu) &= \xi^\nu\partial_\nu(\delta_{\vec{\eta}}A^a_\mu) + (\partial_\mu\xi^\nu)(\delta_{\vec{\eta}}A^a_\nu)\\ &= -\xi^\nu\partial_\nu\big(\partial_\mu\eta^a + f^{abc}A^b_\mu\eta^c\big) - (\partial_\mu\xi^\nu)\big(\partial_\nu\eta^a + f^{abc}A^b_\nu\eta^c\big).\end{aligned} \tag{16}$$

We subtract (16) from (15) to get the commutator of the transformations, which is

$$\begin{aligned}\big[\delta_\xi,\delta_{\vec{\eta}}\big]A^a_\mu &= \xi^\nu\partial_\nu\partial_\mu\eta^a + f^{abc}\xi^\nu(\partial_\nu A^b_\mu)\eta^c - f^{abc}\eta^c\xi^\nu(\partial_\nu A^b_\mu)\\ &+f^{abc}\xi^\nu A^b_\mu(\partial_\nu\eta^c) - f^{abc}\eta^c(\partial_\mu\xi^\nu)A^b_\nu + (\partial_\mu\xi^\nu)(\partial_\nu\eta^a) + f^{abc}(\partial_\mu\xi^\nu)A^b_\nu\eta^c.\end{aligned} \tag{17}$$

The second, third, fifth and seventh terms on the right hand side of (17) cancel out. After applying the Liebniz rule to the first term of (17), we are left with

$$\begin{aligned}&\partial_\mu(\xi^\nu\partial_\nu\eta^a) - (\partial_\mu\xi^\nu)(\partial_\nu\eta^a)\\ &+f^{abc}A^b_\mu(\xi^\nu\partial_\nu\eta^c) + (\partial_\mu\xi^\nu)(\partial_\nu\eta^a) = D_\mu(\xi^\nu\partial_\nu\eta^a).\end{aligned} \tag{18}$$

So the final result is

$$\big[\delta_\xi,\delta_{\vec{\eta}}\big]A^a_\mu = -\delta_{[\delta_\xi,\vec{\eta}]}A^a_\mu, \tag{19}$$

namely that the commutator of a spacetime diffeomorphism with a $SO(3,C)$ gauge transformation acting on A^a_μ is a $SO(3,C)$ gauge transformation acting on A^a_μ.

2.3 The subalgebra of spacetime diffeomorphisms

Lastly, we must check that spacetime diffeomorphisms form a subalgebra of $SO(3,C) * Diff$ on the connection A^a_μ. A diffeomorphism parametrized by ζ followed by a diffeomorphism parametrized by ξ is given by

$$\begin{aligned}\delta_\xi\delta_\zeta A^a_\mu &= \zeta^\nu\partial_\nu(\delta_\xi A^a_\mu) + (\partial_\mu\zeta^a)(\delta_\xi A^a_\nu)\\ &= \zeta^\nu\partial_\nu\big(\xi^\sigma\partial_\sigma A^a_\mu + (\partial_\mu\xi^\sigma)A^a_\sigma\big) + (\partial_\mu\zeta^\nu)\big(\xi^\sigma\partial_\sigma A^a_\nu + (\partial_\nu\xi^\sigma)A^a_\sigma\big)\\ &= \zeta^\nu(\partial_\nu\xi^\sigma)\partial_\sigma A^a_\mu + \zeta^\nu\xi^\sigma\partial_\nu\partial_\sigma A^a_\mu + \zeta^\nu(\partial_\nu\partial_\mu\xi^\sigma)A^a_\sigma\\ &+\zeta^\nu(\partial_\mu\xi^\sigma)\partial_\nu A^a_\sigma + (\partial_\mu\zeta^\nu)\xi^\sigma\partial_\sigma A^a_\nu + (\partial_\mu\zeta^\nu)(\partial_\nu\xi^\sigma)A^a_\sigma.\end{aligned} \tag{20}$$

The result with the vector fields interchanged is given by

$$\delta_\zeta\delta_\xi A^a_\mu = \xi^\nu(\partial_\nu\zeta^\sigma)\partial_\sigma A^a_\mu + \xi^\nu\zeta^\sigma\partial_\nu\partial_\sigma A^a_\mu + \xi^\nu(\partial_\nu\partial_\mu\zeta^\sigma)A^a_\sigma$$
$$+\xi^\nu(\partial_\mu\zeta^\sigma)\partial_\nu A^a_\sigma + (\partial_\mu\xi^\nu)\zeta^\sigma\partial_\sigma A^a_\nu + (\partial_\mu\xi^\nu)(\partial_\nu\zeta^\sigma)A^a_\sigma. \qquad (21)$$

Subtracting (21) from (20), on the right hand side the second terms directly cancel and the fourth and fifth terms cross-cancel. Application of the Liebniz rule to the third term leads to some more cancellations

$$\big[\delta_\xi,\delta_\zeta\big]A^a_\mu = \big(\zeta^\nu\partial_\nu\xi^\sigma - \xi^\nu\partial_\nu\zeta^\sigma\big)\partial_\sigma A^a_\mu$$
$$+\partial_\mu\big(\zeta^\nu\partial_\nu\xi^\sigma - \xi^\nu\partial_\nu\zeta^\sigma\big)A^a_\sigma$$
$$-(\partial_\mu\zeta^\nu)(\partial_\nu\zeta^\sigma)A^a_\sigma + (\partial_\mu\zeta^\nu)(\partial_\nu\xi^\sigma)A^a_\sigma + (\partial_\mu\xi^\nu)(\partial_\nu\zeta^\sigma)A^a_\sigma - (\partial_\mu\xi^\nu)(\partial_\nu\zeta^\sigma)A^a_\sigma, \qquad (22)$$

which annihilate all four terms on the bottom line of (22). The result is is an off-shell closure of the algebra in congruity with (3)

$$\big[\delta_\xi,\delta_\zeta\big]A^a_\mu = -\delta_{[\xi,\zeta]}A^a_\mu, \qquad (23)$$

which shows that spacetime diffeomorphisms are realizable as a symmetry group of the connection A^a_μ.

3 The Ashtekar action

Having shown that the connection A^a_μ forms an off-shell representation of $SO(3,C)*Diff$ symmetry, let us see to what extent this can be implemented at the canonical level. In the canonical formulation one performs a 3+1 splitting of the initially convariant dynamical variables into spatial and temporal parts. For a prototype action based on the connection A^a_μ let us use the Ashtekar formulation of GR [4], which is given by

$$I_{Ash} = \int dt\int_\Sigma d^3x\Big[\widetilde{\sigma}^i_a\dot{A}^a_i + A^a_0G_a - N^iH_i - \frac{i}{2}\underline{N}H\Big]. \qquad (24)$$

This is a totally constrained system with $(A^a_0, N^i, \underline{N})$, respectively the $SO(3,C)$ rotation angle A^a_0, the shift vector N^i and the densitized lapse function $\underline{N} = N(\det\widetilde{\sigma})^{-1/2}$ as auxilliary fields. The phase space variables are the self-dual Ashtekar connection A^a_i and the densitized triad $\widetilde{\sigma}^i_a$. These variables define fundamental Poisson brackets

$$\{A^a_i(x,t),\widetilde{\sigma}^j_b(y,t)\} = \delta^a_n\delta^j_i\delta^{(3)}(x,y);\quad \{A^a_i,A^b_j\} = \{\widetilde{\sigma}^i_a,\widetilde{\sigma}^j_b\} = 0, \qquad (25)$$

which induce the following Poisson brackets between any two smooth phase space functions f and g given by

$$\{f,g\} = \int_\Sigma d^3y\Big[\frac{\delta f}{\delta A^a_i(y)}\frac{\delta g}{\delta\widetilde{\sigma}^i_a(y)} - \frac{\delta g}{\delta A^a_i(y)}\frac{\delta f}{\delta\widetilde{\sigma}^i_a(y)}\Big]. \qquad (26)$$

The constraints in (24) smearing the auxilliary fields are (G_a, H_i, H) the Gauss' law, vector and Hamiltonian constraints

$$G_a = D_i\tilde{\sigma}^i_a; \quad H_i = \epsilon_{ijk}\tilde{\sigma}^j_a B^k_a; \quad H = \epsilon_{ijk}\epsilon^{abc}\tilde{\sigma}^i_a\tilde{\sigma}^j_b\Big(\frac{\Lambda}{3}\tilde{\sigma}^k_c + B^k_c\Big). \tag{27}$$

Choosing g in (26) to be one of the constraints (27) with f as one of the dynamical variables, one can determine the variation of the latter under the transformations generated by the constraints. Under transformations generated by the Gauss' law constraint, the 'small' gauge transformations, the Ashtekar variables transform as

$$\delta_{\vec{\eta}} A^a_i = -D_i\eta^a; \quad \delta_{\vec{\eta}}\tilde{\sigma}^i_a = -f_{abc}\tilde{\sigma}^i_b\eta^c. \tag{28}$$

Comparison of (28) with (8) shows at least that the canonical structure of (24) preserves the correct transformation properties of the spatial part of A^a_μ. The magnetic field B^i_a should transform as a covariant $SO(3,C)$ vector under gauge transformations

$$\delta_{\vec{\eta}} B^i_e = \epsilon^{ijk} D_j(\delta A^e_k) = -\epsilon^{ijk} D_j D_k \eta^e = -f_{efg} B^i_f \eta^g, \tag{29}$$

which is indeed the case as expected. Under transformations generated by the vector constraint, the Ashtekar variables transform as

$$\delta_{\vec{N}} A^a_i = \epsilon_{ijk} B^j_a N^k; \quad \delta_{\vec{N}}\tilde{\sigma}^i_a = \tilde{\sigma}^j_a \partial_j N^i - \partial_j(N^j\tilde{\sigma}^i_a) - f_{abc}(N^j A^b_j)\tilde{\sigma}^i_c. \tag{30}$$

Comparison of (30) with (23) shows that the Ashtekar vector constraint does not generate pure spatial diffeomorphisms, but also includes a gauge transformation with field-dependent parameter $N^i A^a_i$. So far, the spatial components of the Ashtekar variables exhibit the expected transformation properties with respect to $diff$ implied by the Poisson brackets (26). This leaves remaining the 'temporal' diffeomorphisms, which can be seen as the Hamilton's equations of motion.

The evolution equations of the Ashtekar variables generated purely by the Hamiltonian constraint are given by

$$\begin{aligned} \delta_N A^a_i &= -i\underline{N}\big(\Lambda(\det\tilde{\sigma})(\tilde{\sigma}^{-1})^a_i + \epsilon_{ijk}\epsilon^{abc}\tilde{\sigma}^j_b B^k_c\big); \\ \delta_N\tilde{\sigma}^i_a &= -iD^{ni}_{ca}\big(N\sqrt{\det\tilde{\sigma}}(\tilde{\sigma}^{-1})^c_n\big). \end{aligned} \tag{31}$$

The connection has acquired momentum space dependence under Hamiltonian evolution, which is no longer consistent with (9). We say that the 4-dimensional diffeomorphism symmetry is no longer manifestly preserved by the Poisson brackets (26). The constraints algebra for (27)

reads [5]

$$\begin{aligned}
\{\vec{H}[\vec{N}], \vec{H}[\vec{M}]\} &= H_k\big[N^i\partial^k M_i - M^i\partial^k N_i\big];\\
\{\vec{H}[N], G_a[\theta^a]\} &= G_a[N^i\partial_i\theta^a];\\
\{G_a[\theta^a], G_b[\lambda^b]\} &= G_a\big[f^a_{bc}\theta^b\lambda^c\big];\\
\{H[\underline{N}], \vec{H}[\vec{N}]\} &= H[N^i\partial_i\underline{N}]\\
\{H[\underline{N}], G_a(\theta^a)\} &= 0;\\
\big[H(\underline{N}), H[\underline{M}]\big] &= H_i[\big(\underline{N}\partial_j\underline{M} - \underline{M}\partial_j\underline{N}\big)H^{ij}]
\end{aligned} \tag{32}$$

which closes. The algebra (32) is first class, and therefore the theory based on (24) is in this sense Dirac consistent.

Still, comparison of (32) with (2) reveals the same differences pointed out in the introduction, regarding the temporal parts of spacetime diffeomorphisms. In particular, the bracket between two Hamiltonian constraints is a diffeomorphism (and not a Hamiltonian) constraint. The suggestion is that that for the theory (24), spacetime covariance at the canonical level has either been broken or is not manifest. This leaves open the question of whether a formulation of GR exists where two Hamiltonian constraints commute into a Hamiltonian constraint as suggested by (5).[1]

3.1 Comparison with the covariant form

Comparison of (32) with (10) with reveals the aforementioned differences with respect to the temporal components of $Diff$ as shown in the introduction. This suggests that the manifest covariance has been broken, since (24) at the canonical level does not exhibit the same symmetry as the basic fields A^a_μ. We will nevertheless still complete the analysis of (10) with respect to gauge transformations for purposes of comparison with (32). From (19) the commutator of two gauge transformations is given by

$$\big[\delta_{\vec{\eta}}, \delta_{\vec{\theta}}\big] = -\delta_{[\vec{\eta},\vec{\theta}]}, \tag{33}$$

which is consistent with (32). For the result of a spacetime diffeomorphism with a gauge transformation we must revert to (19)

$$\big[\xi^\mu\partial_\mu, \delta_{\vec{\theta}}\big] = -\delta_{L_\xi\theta}. \tag{34}$$

To compare (34) with (32) we must decompose the spacetime diffeomorphisms into a purely spatial part and a purely temporal part. The commutator with the spatial part is given by

$$\big[N^i\partial_i, \delta_{\vec{\theta}}\big] = -\delta_{L_N\vec{\theta}}, \tag{35}$$

[1] We will provide a proposition of such an action in the discussion section, relegating its comparison against (5) for future research.

which is consistent with (32). This shows that the Lie algebra of $SO(3,C) * diff$ is correctly realized by (24) at the canonical level. For the commutator of a gauge transformation and a temporal diffeomorphism (34) yields

$$\left[N\partial_0, \delta_{\vec{\theta}}\right] = -\delta_{N\dot{\theta}}, \tag{36}$$

which is consistent with (32) only for gauge transformations which are independent of time. This is quite analogous to (4) in the introduction.

4 Transformation properties of the CDJ matrix

We have seen, while the 4-D connection A^a_μ forms a representation of $SO(3) * Diff$ symmetry, that only the spatial part of this symmetry is preserved at the canonical level in the Ashtekar formalism (24) which uses A^a_μ as one of the dynamical variables. We will introduce a new quantity $\Psi_{ae} \in SO(3,C) \otimes SO(3,C)$, which in conjunction with A^a_μ will form the dynamical variables for the instanton representation of GR which we will propose in the discussion section. The matrix Ψ_{ae} is known as the CDJ matrix, named after Capovilla, Dell and Jacobson [8].

Consider the following change of variables

$$\widetilde{\sigma}^i_a = \Psi_{ae} B^i_e;\ (\det\Psi) \neq 0,\ (\det B) \neq 0. \tag{37}$$

We will verify that Ψ_{ae} forms an off-shell representation of the $SO(3,C) * diff$ algebra, relegating the temporal part as a direction of future research. First, let us determine the transformation properties of Ψ_{ae} predicted by this algebra. The connection A^a_i and densitized triad $\widetilde{\sigma}^i_a$ under spatial diffeomorphisms $diff$ generated by the vector N^i should transform respectively as covariant and contravariant 3-vectors

$$\delta_{\vec{N}} A^a_i = N^j \partial_j A^a_i + (\partial_i N^j) A^a_j;\ \ \delta_{\vec{N}} \widetilde{\sigma}^j_e = N^k \partial_k \widetilde{\sigma}^j_e - \widetilde{\sigma}^i_e (\partial_i N^j). \tag{38}$$

The first equation of (38) can also be written as

$$\delta_{\vec{N}} A^a_k = D_k (N^m A^a_m) + \epsilon_{klm} B^l_a N^m. \tag{39}$$

Comparison of (39) and the second equation of (38) with (30) shows that the Ashtekar vector constraint generates spatial diffeomorphisms combined with a gauge transformation with field-dependent parameter $N^i A^a_i$. Under $diff$, the magentic field B^i_a transforms as

$$\begin{aligned} \delta_{\vec{N}} B^i_a = \epsilon^{ijk} D_j (\delta A^a_k) &= \epsilon^{ijk} D_j D_k (N^m A^a_m) + \epsilon^{ijk} \epsilon_{lmk} D_j (N^m B^l_a) \\ &= f^{abc} B^i_b (N^m A^c_m) + \left(\delta^i_l \delta^j_m - \delta^i_m \delta^j_l\right) D_j (N^m B^l_a), \end{aligned} \tag{40}$$

were we have used (39), the relation $\delta B^i_a = \epsilon^{ijk} D_j(\delta A^a_k)$, as well as the definition of curvature as the commutator of two covariant derivatives. Continuing from (40), we have

$$\delta_{\vec{N}} B^i_a = f^{abc} B^i_b (N^m A^c_m) + (\partial_m N^m) B^i_a$$
$$+ N^m \partial_m B^i_a + N^m f_{abc} A^b_m B^i_c - (\partial_l N^i) B^l_a - N^i D_l B^l_a. \quad (41)$$

Using the Bianchi identity $D_l B^l_a = 0$ and the fact that the first and fourth terms on the right hand side of (41) cancel, we are left with

$$\delta_{\vec{N}} B^i_a = \partial_m (N^m B^i_a) - (\partial_l N^i) B^l_a. \quad (42)$$

Comparison with the second equation of (38) shows up to a term proportional to $\partial_m N^m$ that equation (42) is consistent with the transformation properties expected of a contravariant 3-vector.

Having written down explicitly the transformations of the Ashtekar variables under $SO(3,C) * diff$, we will now use this to deduce the transformation properties of Ψ_{ae} using (37). Starting with the $SO(3,C)$ gauge transformations, the variation of (37) is given by

$$\delta_{\vec{\eta}} \tilde{\sigma}^i_a = \Psi_{ae} (\delta_{\vec{\eta}} B^i_e) + (\delta_{\vec{\eta}} \Psi_{ae}) B^i_e, \quad (43)$$

where we have used the Liebniz rule. Using (29), this yields the condition

$$-f_{abc} \tilde{\sigma}^i_b \eta^c = \Psi_{ae} (-f_{ebc} B^i_b \eta^c) + (\delta_{\vec{\eta}} \Psi_{ae}) B^i_e, \quad (44)$$

where we have used the Liebniz rule. Using (29) and (42) we have after relabelling indices $e \leftrightarrow g$ and $b \leftrightarrow e$ on the second term

$$-f_{abc} \Psi_{be} \eta^c B^i_e = -\Psi_{ag} f_{gec} B^i_e \eta^c + (\delta_{\vec{\eta}} \Psi_{ae}) B^i_e. \quad (45)$$

Assuming the nondegeneracy condition for B^i_e, then (45) gives us the transformation properties of Ψ_{ae}

$$\delta_{\vec{\eta}} \Psi_{ae} = -\big(f_{abc} \Psi_{be} + \Psi_{ab} f_{ebc}\big) \eta^c. \quad (46)$$

The result is that Ψ_{ae} transforms as a second-rank $SO(3,C)$ tensor under gauge transformations, which seems to makes sense. Next, we would like to determine the transformation properties of Ψ_{ae} under spatial diffeomorphisms. Note that we have

$$\delta_{\vec{N}} \tilde{\sigma}^i_a = \Psi_{ae} (\delta_{\vec{N}} B^i_e) + (\delta_{\vec{N}} \Psi_{ae}) B^i_e, \quad (47)$$

the variation of (37) under spatial diffeomorphisms. Using (38), we have

$$N^m \partial_m \tilde{\sigma}^i_a - (\partial_m N^i) \tilde{\sigma}^m_a = (\delta_{\vec{N}} \Psi_{ae}) B^i_e + \Psi_{ae} \partial_m (N^m B^i_e) - (\partial_m N^i) \Psi_{ae} B^m_e. \quad (48)$$

The second term on the left hand side of (48) cancels the last term on the right hand side, yielding

$$N^m \partial_m \widetilde{\sigma}^i_a = (\delta_{\vec{N}} \Psi_{ae}) B^i_e + \Psi_{ae} \partial_m (N^m B^i_e). \tag{49}$$

Substituting (37) and expanding the derivatives, we have

$$\begin{aligned} N^m \partial_m (\Psi_{ae} B^i_e) &= (\delta_{\vec{N}} \Psi_{ae}) B^i_e + \Psi_{ae} \partial_m (N^m B^i_e) \\ &\longrightarrow N^m (\partial_m \Psi_{ae}) B^i_e + N^m \Psi_{ae} \partial_m B^i_e \\ = (\delta_{\vec{N}} \Psi_{ae}) B^i_e &+ \Psi_{ae} (\partial_m N^m) B^i_e + \Psi_{ae} N^m (\partial_m B^i_e). \end{aligned} \tag{50}$$

After cancellation of various terms in (50), we obtain

$$\delta_{\vec{N}} \Psi_{ae} = N^m \partial_m \Psi_{ae} - (\partial_m N^m) \Psi_{ae}. \tag{51}$$

Equations (46) and (51) provide the desired transformation properties of Ψ_{ae} under $SO(3,C) * diff$. We will now prove that Ψ_{ae} forms an off-shell representation of the $SO(3,C) * diff$ algebra, which we have relegated to the appendix. The result is shown in the discussion section.

5 Discussion

This paper has proposed variables (A^a_μ, Ψ_{ae}), with a view toward restoring the full covariance of general relativity under spacetime diffeomorphisms. The connection A^a_μ forms an off-shell representation of $SO(3,C) * Diff$ symmetry, and in the case of Ψ_{ae} we have proven only $SO(3,C) * diff$ symmetry, which does not include the temporal components of $Diff$. The algebra is given by

$$\begin{aligned} [\delta_{\vec{\theta}}, \delta_{\vec{\eta}}] A^a_\mu = -\delta_{\vec{\theta} \times \vec{\eta}} A^a_\mu; \quad [\delta_{\vec{\theta}}, \delta_{\vec{\eta}}] \Psi_{ae} = -\delta_{\vec{\theta} \times \vec{\eta}} \Psi_{ae} \\ [\delta_\xi, \delta_{\vec{\eta}}] A^a_\mu = -\delta_{\delta_{L_\xi} \vec{\eta}} A^a_\mu; \quad [\delta_{\vec{N}}, \delta_{\vec{\eta}}] \Psi_{ae} = -\delta_{L_{\vec{N}} \vec{\eta}} \Psi_{ae} \\ [\delta_\xi, \delta_\zeta] A^a_\mu = -\delta_{[\xi,\zeta]} A^a_\mu; \quad [\delta_{\vec{M}}, \delta_{\vec{N}}] \Psi_{ae} = -\delta_{[\vec{N},\vec{M}]} \Psi_{ae}. \end{aligned} \tag{52}$$

We have shown that the Ashtekar formulation of general relativity defined on the phase space $\Omega_{Ash} = (\widetilde{\sigma}^i_a, A^a_i)$ implements only the $SO(3,C) * diff$ part of the algebra, and only for spatial diffeomorphisms and $SO(3,C)$ gauge transformations which are independent of time. This suggests that the $SO(3,C) * Diff$ symmetry of the connection A^a_μ has been broken at the canonical level. A question which we propose for future research is whether this symmetry can to some extent be restored at the canonical level using an alternate formulation of GR. We will propose a new action

$$\begin{aligned} I_{Inst} = \int dt \int_\Sigma d^3x \Big(\Psi_{ae} B^i_e \dot{A}^a_i + A^a_0 B^i_e D_i \Psi_{ae} \\ + \epsilon_{ijk} N^i B^j_a B^k_e \Psi_{ae} - iN (\det B)^{1/2} \sqrt{\det \Psi} (\Lambda + \mathrm{tr} \Psi^{-1}) \Big), \end{aligned} \tag{53}$$

which can be obtained by substitution (37) into the action (24). Using the definition $F^a_{0i} = \dot{A}^a_i - D_i A^a_0$ for the temporal component of the curvature, and using the identity

$$\frac{1}{2}\int dt \int_\Sigma \Psi_{(ae)}\epsilon^{ijk}F^e_{jk}F^a_{0i} = \frac{1}{8}\int_M d^4x \Psi_{ae}F^a_{\mu\nu}F^e_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}, \tag{54}$$

then upon separation of Ψ_{ae} into symmetric and antisymmetric parts we can write the action (53) as

$$I_{Inst} = \int dt \int_\Sigma d^3x \big(B^i_{[e}F^{a]}_{0i} + \epsilon_{ijk}N^i B^j_a B^k_e\big)\Psi_{ae}$$
$$+ \int_M d^4x \left[\frac{1}{8}\Psi_{ae}F^a_{\mu\nu}F^e_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} - iN(\det B)^{1/2}\sqrt{\det\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big)\right]. \tag{55}$$

As a direction of future research we will check the algebra of constraints implied by (55) to examine to what extent the $SO(3,C)*Diff$ symmetry can be restored. Since (5) suggests that temporal diffeomorphisms should form their own subalgebra, an additional area of research will be to investigate whether this can be deduced from (53). It is shown in [9], using a reduced version of (53), that the Hamiltonian constraint does form a subalgebra, albeit with structure functions and not structure constants.

6 Appendix A: Closure of the $SO(3,C) * diff$ algebra on Ψ_{ae}

6.1 Closure under spatial diffeomorphisms

Having determined the transformation properties of Ψ_{ae}, we will now check for off-shell closure of the constraints algebra on Ψ_{ae}, seen as a dynamical variable. Let us first examine the effect of two consecutive diffeomorphisms

$$\delta_{\vec{M}}\delta_{\vec{N}}\Psi_{ae} = N^m\partial_m(\delta_{\vec{M}}\Psi_{ae}) - (\partial_m N^m)\delta_{\vec{M}}\Psi_{ae}$$
$$= N^m\partial_m\Big(M^n\partial_n\Psi_{ae} - (\partial_n M^n)\Psi_{ae}\Big) - (\partial_m N^m)\Big(M^n\partial_n\Psi_{ae} - (\partial_n M^n)\Psi_{ae}\Big)$$
$$= N^m(\partial_m M^n)\partial_n\Psi_{ae} + N^m M^n\partial_m\partial_n\Psi_{ae} - (\partial_m N^m)M^n\partial_n\Psi_{ae} + (\partial_m N^m)(\partial_n M^n)\Psi_{ae}$$
$$-N^m(\partial_m\partial_n M^n)\Psi_{ae} - N^m(\partial_n M^n)\partial_m\Psi_{ae}, \tag{56}$$

where we have used (51). After cancellation of various terms and subtracting the result with $\vec{M}$ and $\vec{N}$ interchanged, we obtain

$$\big(\delta_{\vec{M}}\delta_{\vec{N}} - \delta_{\vec{N}}\delta_{\vec{M}}\big)\Psi_{ae} = (L_{\vec{N}}\vec{M})^n\partial_n\Psi_{ae} + \big(M^n\partial_m\partial_n N^n - N^m\partial_m\partial_n M^n\big)\Psi_{ae}$$
$$= (L_{\vec{N}}\vec{M})^n\partial_n\Psi_{ae} - \partial_n(L_{\vec{N}}\vec{M})^n\Psi_{ae} \tag{57}$$

where $(L_{\vec{N}}\vec{M})^n = N^m\partial_m M^n - M^m\partial_m N^n$ is the n^{th} component of the Lie derivative of M^i along N^i. The result is

$$\big[\delta_{\vec{M}}, \delta_{\vec{N}}\big]\Psi_{ae} = -\delta_{[\vec{N},\vec{M}]}\Psi_{ae}, \tag{58}$$

namely that the commutator of two spatial diffeomorphisms is also a spatial diffeomorphism. The spatial diffeomorphisms form a closed algebra on Ψ_{ae}, which is consistent with (3) and with the purely spatial part of (32).

6.2 Closure under mixed transformations

Next we move on to the commutator of a spatial diffeomorphism with a gauge transformation. Using (46) and (51), this is given by

$$\begin{aligned}\big[\delta_{\vec{N}}, \delta_{\vec{\eta}}\big]\Psi_{ae} &= \delta_{\vec{N}}(\delta_{\vec{\eta}}\Psi_{ae}) - \delta_{\vec{\eta}}(\delta_{\vec{N}}\Psi_{ae}) \\ &= -\big(f_{abc}\delta_{\vec{N}}\Psi_{be} + f_{ebc}\delta_{\vec{N}}\Psi_{ab}\big)\eta^c - \big(N^m\partial_m(\delta_{\vec{\eta}}\Psi_{ae}) - (\partial_m N^m)\delta_{\vec{\eta}}\Psi_{ae}\big) \\ &= -f_{abc}\big(N^m\partial_m\Psi_{be} - (\partial_m N^m)\Psi_{be}\big)\eta^c - f_{ebc}\big(N^m\partial_m\Psi_{ab} - (\partial_m N^m)\Psi_{ab}\big)\eta^c \\ &\quad + \big(N^m\partial_m - (\partial_m N^m)\big)\big(f_{abc}\Psi_{be} + f_{ebc}\Psi_{ab}\big)\eta^c. \end{aligned}\tag{59}$$

Expanding the terms, we obtain

$$\begin{aligned}&-f_{abc}N^m(\partial_m\Psi_{be})\eta^c + f_{abc}(\partial_m N^m)\Psi_{be}\eta^c - f_{ebc}N^m(\partial_m\Psi_{ab})\eta^c + f_{ebc}(\partial_m N^m)\Psi_{ab}\eta^c \\ &+f_{abc}N^m(\partial_m\Psi_{be})\eta^c + f_{abc}N^m\Psi_{be}(\partial_m\eta^c) + f_{ebc}N^m(\partial_m\Psi_{ab})\eta^c + f_{ebc}N^m\Psi_{ab}(\partial_m\eta^c) \\ &\quad -f_{abc}(\partial_m N^m)\Psi_{be}\eta^c - f_{ebc}(\partial_m N^m)\Psi_{ab}\eta^c = (N^m\partial_m\eta^c)\big(f_{abc}\Psi_{be} + f_{ebc}\Psi_{ab}\big)\end{aligned}\tag{60}$$

which leads to cancellation of several terms. The result is that

$$\big[\delta_{\vec{N}}, \delta_{\vec{\eta}}\big]\Psi_{ae} = -\delta_{L_{\vec{N}}\vec{\eta}}\Psi_{ae}, \tag{61}$$

namely that the commutator of a gauge transformation and a spatial diffeomorphism is a gauge transformation, which is also consistent with (32).

6.3 Closure under $SO(3,C)$ gauge transformations

Next we move on to the commutator of two gauge transformations. Two successive gauge transformations yield

$$\delta_{\vec{\theta}}\delta_{\vec{\eta}}\Psi_{ae} = -\big(f_{abc}\delta_{\vec{\theta}}\Psi_{be} + f_{ebc}\delta_{\vec{\theta}}\Psi_{ab}\big)\eta^c. \tag{62}$$

Expanding this out and using (46), we obtain

$$\begin{aligned}&f_{abc}\big(f_{bfd}\Psi_{fe} + f_{efd}\Psi_{bf}\big)\theta^d\eta^c + f_{bec}\big(f_{afd}\Psi_{fb} + f_{bfd}\Psi_{af}\big)\theta^d\eta^c \\ &= \big(\delta_{ad}\delta_{cf} - \delta_{af}\delta_{cd}\big)\Psi_{fe} + f_{abc}f_{efd}\Psi_{bf} + f_{ebc}f_{afd}\Psi_{fb} + \big(\delta_{ed}\delta_{cf} - \delta_{ef}\delta_{cd}\big)\Psi_{af}.\end{aligned}\tag{63}$$

Expanding (63) and subtracting the result with $\vec{\theta}$ and $\vec{\eta}$ interchanged, we obtain

$$\big[\delta_{\vec{\theta}}, \delta_{\vec{\eta}}\big]\Psi_{ae} = \Big(\delta_{ad}\Psi_{ce} - \delta_{cd}\Psi_{ae} + \delta_{ed}\Psi_{ac} - \delta_{cd}\Psi_{ae} + f_{abc}f_{efd}\Psi_{bf} + f_{ebc}f_{afd}\Psi_{fb}\Big)\theta^d\eta^c$$
$$+\Big(-\delta_{ac}\Psi_{de} + \delta_{dc}\Psi_{ae} - \delta_{ec}\Psi_{ad} + \delta_{dc}\Psi_{ae} - f_{abd}f_{efc}\Psi_{bf} - f_{ebd}f_{afc}\Psi_{fb}\Big)\eta^c\theta^d = A + B, \quad (64)$$

where we have relabelled $c \leftrightarrow d$ in the second line. We have also defined

$$A = \big(\delta_{ad}\Psi_{ce} - \delta_{ac}\Psi_{de} + \delta_{ed}\Psi_{ac} - \delta_{ec}\Psi_{ad}\big)\theta^d\eta^c;$$
$$B = \Big(f_{abc}f_{efd}\Psi_{bf} + f_{ebc}f_{afd}\Psi_{fb} - f_{abd}f_{efc}\Psi_{bf} - f_{ebd}f_{afc}\Psi_{fb}\Big)\theta^d\eta^c. \quad (65)$$

Note that $B = 0$, which can easiest be seen by relabelling $f \leftrightarrow b$ on the second and fourth terms of (65). Hence it remains to show that A in (46) is a gauge transformation. This can be written as

$$A = \delta_{ad}\Psi_{ce}(\theta^d\eta^c - \theta^c\eta^d) + \delta_{ed}\Psi_{ac}(\theta^d\eta^c - \theta^c\eta^d) = \big(\delta_{ad}\Psi_{ce} + \delta_{ed}\Psi_{ac}\big)(\theta^d\eta^c - \theta^c\eta^d). \quad (66)$$

Defining $\theta^d\eta^c - \theta^c\eta^d = \epsilon^{dcg}q^g$, where $q^g = \epsilon^{ga'e'}\theta^{a'}\eta^{e'}$, then we have

$$A = \big(\delta_{ad}\Psi_{ce} + \delta_{ed}\Psi_{ac}\big)\epsilon_{dcg}q^g = \big(f_{acg}\Psi_{ce} + f_{ecg}\Psi_{ac}\big)q^g. \quad (67)$$

The result is

$$\big[\delta_{\vec{\theta}}, \delta_{\vec{\eta}}\big]\Psi_{ae} = -\delta_{\vec{\theta}\times\vec{\eta}}\Psi_{ae}, \quad (68)$$

namely that the commutator of two gauge transformations is a gauge transformation. Comparison of the results of (58), (61) and (68) with (32) shows that the constraints algebra for kinematic transformations is preserved under the change of variables (37). For the temporal part of the transformations of Ψ_{ae}, we will defer treatment to a separate paper.

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CONFUSION IN THERMODYNAMICS

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Abstract

For a long time now, confusion has existed in the minds of many over the meaning of various concepts in thermodynamics. Recently, this point has been brought to people's attention by two articles appearing on the well-known archive (arXiv) web site. The content of these two pieces serves to illustrate many of the problems and has occasioned the construction of this answer to at least some of them. The position of the axiom proposed by Carathéodory is central in this matter and here its position is clarified and secured within the framework of thermodynamics. In particular, its relation to the First Law is examined and justified.

Keywords: Thermodynamic concepts, second law, Carathéodory.

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Introduction.

There is little doubt that, although thermodynamics is based on phenomena and, possibly more importantly, experiences with which virtually everyone is familiar, confusion does arise in the minds of many when it comes to understanding. However, as was stated on the dust cover of Landsberg's first book on the subject [1],

'Thermodynamics is among the most abstract branches of physics'.

Considering the beginnings of the subject were so firmly rooted in so practical a subject as the theory of heat engines, this appears to be an astonishing statement. However, a little delving into the theory shows it to be a fairly accurate assessment and probably indicates one of the reasons for the confusion arising in so many minds. Added to this thought, it is often forgotten that, when considering theories of physics in general, the discussion concerns theories which purport to describe perceived physical phenomena. As such, it should be remembered always that it is the *physics* which is all-important; it is the physical observations which must always provide the impetus in any subsequent investigations. In all this, the mathematics is merely a tool; a very important tool, but nonetheless only a tool. Therefore, any deductions made must be checked against observed physical fact. It is the *physics* which must dominate! On the other hand, it often proves difficult to do this as the mathematical theory usually appears abstruse to the uninitiated and frequently serves to add to the mysticism some would wish to attach to their chosen field. In fact, it has been known in some fields for people to pursue highly productive careers doing no more than re-deriving known results in a more abstruse mathematical formulation. There is little doubt that this is one factor leading to confusion in the minds of many in various areas of science, of which thermodynamics is but one, and in this paper we clarify some of the misconceptions that have appeared, and continue to appear, in the literature around the various forms of the First and Second Laws of thermodynamics.

The Carathéodory and Kelvin Forms of the Second Law.

The argument employed in a recent posting by Radhakrishnamurty [2] on the well-known Cornell University administered archive site is a perfect example of where this confusion has helped towards a chaotic misunderstanding of some of the basics of thermodynamics. The claims concerned the validity of the proofs of the equivalence of the

Carathéodory and Kelvin forms of the Second Law of thermodynamics. He based his claim on the version of the Second Law which appears in Fermi's book on thermodynamics [3]. As printed, this states that

"A transformation whose only final result is to transform into work heat extracted from a source which is at the same temperature throughout is impossible."

As with all forms of the Second Law, whether old or more modern, it is important to read *all* the words and realise that they *all* form crucial parts of the statement. In the version quoted here, one highly important part is *'whose only final result'*. This means quite simply that, at the end of the procedure, the only end result is the transformation of an amount of heat into work. This means that, at the end of the procedure, the entire system must be back where it started or, in other words, the whole process must have been cyclic. Nowadays, statements of the law usually refer specifically to everything occurring in a cycle. However, in Kelvin's day, as is seen quite clearly from the writings of Tait [4], Kelvin was constantly discussing cycles and probably saw little need to include the word explicitly in his statement of the Second Law. Indeed, in describing the *Fundamental Principles in the Theory of the Motive Power of Heat* in 1851 [5] Kelvin remarked, "Whenever in what follows *the work done* or *the mechanical effect produced* by a thermo-dynamic engine is mentioned without qualification, it must be understood that the mechanical effect produced, either in a non-varying engine, or in a complete cycle, or any number of complete cycles of a periodic engine, is meant". This statement appeared at the start of the paper and sets out very clearly the context in which the rest of it should be understood, but without this qualification any subsequent statement quoted in isolation will be out of context. That context is provided by the cycle, which is central to any discussion of thermodynamic engines. Incidentally, this is probably one of the earliest uses of the word "thermo-dynamics" and shows clearly that the subject arose out of the study of heat engines which were so important for raising and lowering cages in mines and for pumping water out of many of those mines.

What Landsberg showed in his article [6] was that, if Carathéodory's accessibility criterion was untrue, then an amount of heat could be converted completely into work in a cycle, contrary to the Kelvin form of the Second Law. In Landsberg's cycle, although not stated explicitly since such a statement was not necessary, heat is given to the system from a

heat reservoir and, therefore, that heat is transferred from a source at constant temperature. Then, assuming Carathéodory's axiom is not valid, which would imply that, in the neighbourhood of a state, *all* other states would be adiabatically accessible from it, the cycle is completed by returning to the starting point by using an adiabatic process. Hence, the proof that Kelvin's form of the Second Law implies the validity of Carathéodory's axiom.

Of course, some point out that one of the processes here is isothermal and so occurs at a constant temperature, whereas the other is adiabatic and so is accompanied by a temperature change. Hence, the above mentioned cycle is impossible and so the proof invalid. This is simply not true, as Clausius' theorem demonstrates. Clausius showed that around a closed cycle the total entropy change is less than, or equal to, zero, which means that any input of heat at a high temperature in order to do work has to be accompanied by an outflow of heat at a lower temperature in order to return the system to its initial state. In other words, the initial state is inaccessible by an adiabatic process from whatever state the system occupies after the input of heat. This is essentially the argument advanced by Landsberg, who made no assumptions about isothermal and adiabatic processes except that, in the first, heat is given to the system at a fixed temperature while, in the second, work is done but no heat is transferred. The Second Law in its more traditional form due to Kelvin then rules out the possibility of such a cycle and the validity of the Carathéodory axiom is verified. There is no contradiction here.

A similar argument is used to see in what sense the reverse is true [7] and it is found that Carathéodory's axiom is more general than the usual form of Kelvin's statement of the Second Law but equivalent to the modified form applicable when negative temperatures are considered. Hence, the claims of Radhakrishnamurty are unfounded and, if you consider the form of Kelvin's statement usually in use today [8] and which was the one used by both Landsberg and Dunning-Davies in their discussions:

"It is impossible to transform an amount of heat completely into work in a cyclic process in the absence of other effects",

you can see immediately that there are no real differences between it and the form Fermi attributes to Kelvin; the main difference is that this later version specifically mentions cycles, while the concept is implied in the

quoted Fermi version. This, of course, is yet another point that Radhakrishnamurty fails to appreciate; he constantly criticises Landsberg for using an 'incorrect' form of Kelvin's statement of the Second Law.

This situation illustrates, once again, how confusion and lack of understanding can creep into people's knowledge of thermodynamics. However, in this particular instance, it is due to a lack of care in reading the separate statements of – in this case – the Second Law. It is always vitally important to read and digest each and every word of such statements. As pointed out earlier, in the form attributed by Fermi to Kelvin, the complete meaning of the words *'whose only final result'* must be absorbed and followed. The more modern wording, given above and used by Landsberg, merely singles out for special mention the fact that everything must occur in a cycle so that, after everything, the entire system ends up in exactly the same position from which it started. Not realising that both forms are actually identical indicates a lack of basic understanding and an inability to read and absorb information in totality. Of course, this also raises grave questions over the process which allowed the original article to appear on the archive site and the need to ask this question is reinforced further by an even more recent posting [9] on the same site.

Carathéodory and the First Law.

The second article refers to the relationship between Carathéodory's axiom and the First Law of thermodynamics, and makes the point that the axiomatic approach is highly abstract and difficult to use. However, Carathéodory's approach is rarely used in its original, highly mathematical form, and this point seems to have been missed by Radhakrishnamurty. Carathéodory was not himself a thermodynamicist, but was invited to make a contribution to the subject by Max Born and while Born himself made contributions, notably via his book *Natural Philosophy of Cause and Chance* [10], the new more abstract approach was clarified and modified by several people, mainly P.T. Landsberg in England, L. A. Turner, F. W. Sears and M. W. Zemansky in the U.S.A., and H. A. Buchdahl in Australia. The end result of their work has seen the virtual abandonment of Carathéodory's axiom from use but the retention of methods originally associated with it, namely the technique for deriving the existence of both an absolute temperature and entropy from the Second Law, as well as the equation representing the Second Law:

$$d'Q = TdS$$

where the symbol *d´* indicates an inexact differential. This is the principal legacy of Carathéodory.

However, in this second article by Radhakrishnamurty [9], it is noted that, if a system undergoes an adiabatic change between two equilibrium states *A* and *B*, the First Law reduces to

$$dU = d'W$$

That is, during the change, work done produces a change in internal energy. The author then considers the case when

$$dU \neq d'W$$

It is here that Radhakrishnamurty exhibits the greatest confusion. Stating correctly that 'state *B* is not reachable by an adiabatic process from state *A* only if $dU \neq d' W'$, he goes on to claim that 'in other words state *B* is not reachable from state *A* by an adiabatic process only if the First Law is violated.'

It is the second sentence here which is totally misleading. The assumption that B is inaccessible from A is nothing more than a contrary assumption to the original, namely that the process linking the two states is adiabatic. The First Law can be written generally as

$$d'Q = dU - d'W$$

Therefore any non-adiabatic process is characterised by a contribution from the term *d´Q* ; the quantity of heat that makes up the difference between the terms *dU* and *d´W*.

The nature of this contribution can be elucidated from a couple of examples of irreversible adiabatic processes. Adiabatic irreversibility is very important in thermodynamics and directly relevant to the discussion on Carathéodory. If Carathéodory is invalid so that all states in the neighbourhood of every other state are accessible by an adiabatic process, then state *A* is accessible from *B* and vice versa. This would mean that all

adiabatic processes are reversible, so the mere fact of an irreversible adiabatic process demonstrates the validity of Carathéodory's axiom. Consider by way of example the free expansion from say *A* to *B*. State *A* cannot be attained from *B* by an adiabatic process as the volume must be decreased and work done on the gas. It is necessary, therefore to extract heat from the system in order to return to state *A*. Therefore state A is not accessible from state B by an adiabatic process.

Other irreversible adiabatic processes have recently been discovered which also illustrate inaccessibility. The authors have recently examined a simple thermodynamic system, the compound piston, by means of computer simulation [11,12]. The adiabatic compound piston, which is essentially a cylinder filled with gas and divided in two by a moveable piston, was described as long ago as 1960 by Callen [13] but without coming to a successful conclusion about the final state. In fact, Callen argued that a frictionless, adiabatic piston would oscillate indefinitely. This is easily demonstrated if Newton's second law of motion is solved with the equation of state for the ideal gas giving the pressure. The authors established a computational simulation in which hard spheres on either side of the piston undergo elastic collisions between themselves and with the piston and showed that, at the kinetic level, if the simulation is started with the piston displaced to one side so that the pressures on either side are unequal, equilibrium is eventually reached. The kinetic energy of the piston is dissipated within the gas by inter-particle collisions which cause a redistribution of energy between the orthogonal degrees of freedom, leading ultimately in equilibrium to equipartition of the internal energy. Detailed simulations enabled investigation of the macroscopic thermodynamics of these irreversible processes and we were able to show that for a limited change in volume the work done is given be the product of the final pressure and the change in volume. A mathematical analysis confirmed this result.

This collisional mechanism of damping does not need to occur on both sides of the piston for damping to occur and we were able to extend the simulations to the case of a cylinder open at one end so that a constant pressure acts against the piston. The work done is identical to the above, being simply the product of the external, or final, pressure, and the volume change. This irreversible process may be incorporated into a cycle as follows. Consider a system in equilibrium so that the internal pressure is

equal to the external pressure. Allow the external pressure to drop so that the piston moves outward and settles at a new position. The pressure and temperature inside the cylinder have both dropped as work has been done against the external pressure. Now let the external pressure increase to its original value so that the piston moves in the opposite direction and settles again at a new position at which the internal and external pressures are equal. It is self-evidently the case, confirmed by simulation, that the volume at the end of this compression cannot be equal to the initial volume otherwise a contradiction occurs. If the volume changes in both directions are equal in magnitude, the work done in compression will be greater than the work done during the expansion owing to the difference in the external pressures. Therefore, the temperature of the gas must be greater than the original temperature and the pressure would therefore also be greater than the original pressure, which is impossible. By similar reasoning the volume change during compression cannot be larger than the volume change during the expansion and it follows that the final volume must lie somewhere between the initial value and the volume after expansion. As the external pressure is equal to the initial pressure it also follows that in order to return the system to its initial state it is only necessary to cool the system to its original temperature.

This sequence of operations just described corresponds to a complete cycle in which the initial state is adiabatically inaccessible from the second state. Moreover, as with the free expansion, heat must be extracted to restore the system to the initial state. If a cycle is conducted in the reverse direction, so that the pressure is initially increased and then dropped to its initial value, one finds a similar result. The work done during the expansion cannot be the same as the work done during compression as this would have the effect of restoring the initial temperature but not the volume. Nor can the work done be greater as the temperature of the gas would be lower than the initial value, leading to the following contradiction; as the pressure is the same as the original pressure, a lower temperature would also imply a smaller volume, but that is impossible as the greater work done against a lower pressure would imply a larger volume change. The *change* in volume on the subsequent expansion must be less than that done during the compression. The temperature will therefore be higher than the original and heat will have to be extracted to restore the initial condition.

These three cycles all demonstrate adiabatic inaccessibility via the existence of irreversible adiabatic processes. Moreover, they all appear to demonstrate that heat must be extracted from a system if a cycle is performed in which one of the stages consists of just such an irreversible process. In fact, this is a general result. As we have mentioned above, Clausius was the first to enunciate the idea that around a closed, irreversible cycle the integral of

$$d'Q/T \leq 0.$$

Adiabatic inaccessibility therefore implies Clausius' theorem.

Conclusion

As acknowledged earlier, thermodynamics is a subject which causes problems for people at all levels of academia. From the outset, it must be acknowledged that one way in which problems are created is through unclear and muddled thinking and writing. This is a problem which has plagued thermodynamics almost from the very birth of the subject. The writings of Clausius are a typical case in point and it might be argued that the foundations of much of the confusion existing today rest in his early articles. This problem has begun to be addressed in a variety of ways [8]. The finding that a kinetic mechanism of damping exists and that the work done in the compound piston is $P_{final}\int dV$ rather than $\int PdV$ has a significance beyond the question of adiabatic inaccessibility and the more modern approach associated with the name of Carathéodory, but that is the primary concern here.

The note to which reference is being made [9] is an excellent example of the confusion that can arise in thermodynamics. This author believes he has shown first that Landsberg's demonstration that Carathéodory's axiom is compatible with Kelvin's statement of the Second law is false and secondly, that the assumption that Carathéodory is valid leads to a violation of the First Law. With regard to the latter, Radhakrishnamurty has simply neglected the contribution of heat to the First Law equation of energy conservation. The conclusion supposedly deduced from the first quoted sentence from the quoted article [9] simply does not follow and Carathéodory's axiom is found to be perfectly compatible with the First Law of Thermodynamics. We have demonstrated this with reference to irreversible adiabatic processes and shown that in order to return to the

original state it is necessary to extract heat from the system. We have shown that Kelvin, Clausius and Carathéodory are all linked, all mutually compatible and none involve a violation of the First Law, which is, in fact, what Radhakrishnamurty has actually demonstrated.

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Addendum.

It might be of interest to note that this article has been occasioned by two postings, both by Radhakrishnamurty [2,9], on the arxiv site. As commented earlier, it is somewhat surprising that the first of these [2] survived that moderation process of the site but it is almost unbelievable that the second article [9] did so as well. It might be noted also that, although published work in a prestigious journal, *Nature*, by Landsberg and Dunning-Davies was criticised in [2], the site had to be vigorously persuaded to post a response by Dunning-Davies. This posting was exactly preceded in the arxiv list by [9] and, in order to respond to these totally separate article, Dunning-Davies and Sands were forced to modify the earlier response by Dunning-Davies to [2]; two separate responses to the two separate postings by Radhakrishnamurty were not allowed.

SOLVING EINSTEIN TWINS' PARADOX

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Abstract

This paper uses the theory of quantized time and length as well as a new model of the internal structure of elementary particles via my expression [1] to solve the Einstein twins' paradox.

Key words: Length quanta, time quanta, paradox, moton

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1-introduction

Each observer in relativity theory has his own measurement of time and this leads to the twins' paradox [2], [3]. Assume one of the twins, A, goes away from earth at a speed of about 98 percent of the speed of light and, after a trip of five years, returns to earth. The other twin, B, remains on earth. Based on the theory of relativity after returning, twin A will find that his brother has aged more than five years. Now again based on relativity theory, if it is assumed that twin A is stationery and B is in motion, a paradox is produced.

2-Solving the paradox

The Einstein twins' paradox can be solved by knowing that the motons have a motion in the elementary particle.[1] To solve the paradox we assume that the motion of each twin is equal to the vector sum of the motons of the same twin. On the other hand, motion is not an abstract motion which is given to a particle, but it exists in the nature of the particles.

We assume twin A consists of the following motons

$$A = \sum_{n=1}^{n=k1} a_n \qquad (1)$$

and the twin B includes the following ones

$$B = \sum_{n=1}^{n=k2} b_n \qquad (2)$$

Then according to the formula [1]

$$V^2 = \left(\frac{l_{1(so1)}}{\Delta T_{1(so1)}} \times \frac{l_{2(so1)}}{\Delta T_{2(so1)}}\right) - \left(\frac{l_{2(so2)}}{\Delta T_{2(so2)}} \times \frac{l_{3(so3)}}{\Delta T_{3(so3)}}\right) \qquad (3)$$

The speed of twin A equals

$$\vec{V}_A^{\,2} = \sum_{n=1}^{n=k1moton} \vec{V}_n \qquad (4)$$

and the speed of twin B equals

$$\vec{V}_B^{\,2} = \sum_{n=1}^{n=k2moton} \vec{V}_n \qquad (5)$$

Formula (3) consists of two parts of which the absolute part is:

$$V_0^{\,2} = \left(\frac{l_{1(so1)}}{\Delta T_{1(so1)}} \times \frac{l_{2(so1)}}{\Delta T_{2(so1)}}\right) = C^2 \qquad (6)$$

$$l_0 = l_1 = l_{2\,(S_{01})} = 1.409 \times 10^{-15}\text{m} \qquad (7)$$

$$\Delta T_0 = \Delta T_1 = \Delta T_{2\,(S_{01})} = 0.47 \times 10^{-23}\text{s} \qquad (8)$$

And the variable part is:

$$\frac{C^2}{\gamma^2} = \left(\frac{l_{2(so2)}}{\Delta T_{2(so2)}} \times \frac{l_{3(so3)}}{\Delta T_{3(so3)}}\right) \qquad (9)$$

$$l_{3\,(S_{02})} = \frac{l_0}{\gamma} \qquad (10)$$

$$\Delta T_{3\,(S_{02})} = \Delta T_0 \gamma \qquad (11)$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}: \text{ Lorentz coefficient} \qquad (12)$$

$$V^2 = C^2 - \frac{C^2}{\gamma^2} \qquad (13)$$

$$l = \frac{l_0}{\gamma} \qquad (14)$$

$$\Delta T = \Delta T_0 \gamma \qquad (15)$$

$$|V| = \sqrt{c^2 - \frac{c^2}{\gamma^2}} \qquad (16)$$

Where V is the relative velocity of the particle

Then we have two parts. One of them has an absolute value (equation 6) and the other one has a variable value (equation 9). The value of equation 9 will be subtracted from the absolute value of equation 6.

For measuring the particle velocity, a system should be used which takes the absolute part of the formula as the basis of its measurement. On the other hand, the device should work at almost zero Kelvin temperature. By considering the

absolute part, the other part of the formula, which is relative, will be measured. By the formula (6) we can always have the reference time and length quanta and by referring to them, the changes in time and length quanta in formula (9) can be obtained. For doing this, a measuring device can be considered for each twin, which receives a specific electromagnetic signal from each twin and analyzes it. To calculate the right value of equation 6 the temperature of this equipment should be at about zero Kelvin. If we assume that the entire trip takes M time quanta, we have two inequalities for the travelled path. "The amount of M obtained from equation 6."

In a test,

$$\sum_{m=1}^{m=M} \Delta T_{0m} = \sum_{n=1}^{n=M} \Delta T_{0n} \qquad (17)$$

Base on the assumption of movement of twin A by 98% speed of light:

$$\vec{V}_m \prec \vec{V}_n \qquad (18)$$

$$\sum_{m=1}^{m=M} \Delta T_{0m} \sum_{n=1}^{n=K2moton} \vec{v}_n \prec \sum_{n=1}^{n=M} \Delta T_{0n} \sum_{n=1}^{n=k1moton} \vec{v}_n \qquad (19)$$

It means the travelled path in a certain time for the twin A is greater than that for twin B. Again, the speed of twin A is always greater than that of twin B and we will have the following inequality:

$$\vec{V}_B \prec \vec{V}_A \prec \vec{C} \qquad (20)$$

It should be noticed that, in relativity the speed of twins related to each other is obtained when one of them is considered stationary and another one is considered mobile. This issue produces a paradox. In this theory [1] each twin can be considered both stationary and mobile and by formula (6) which has an absolute part we can understand which twin has a greater speed.

According to formula 15 the following inequality is obtained:

$$\Delta T_B \prec \Delta T_A \qquad (21)$$

So if the M_A time quanta pass for the twin A, M_B time quanta pass for twin B and the entire process takes M time quanta(according to equation 6) we will have the following inequality:

$$M_A \prec M_B \prec M \qquad (22)$$

M_A and M_B can be obtained from equation 9.
Therefore the time which is passed for twin A is less than twin B. It means the twin A is younger than twin B.

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SUSY & STRONG NUCLEAR GRAVITY IN (120-160) GeV MASS RANGE

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Abstract

It is suggested that, in super symmetry, the fermion and boson mass ratio is $\Psi \cong 2.262218404$ but not unity. Based on strong nuclear gravity and super symmetry in our previous papers it is suggested that: the W boson is the super symmetric boson of the top quark fermion and the charged Higg's boson pair generates the neutralized Z boson. In this paper it is suggested that the charged Higg's fermion, and nuclear charge radius play a crucial role in the emission of the electron in Beta decay. It is noticed that the purpose of the previously proposed 11460 MeV fermion can be obtained with the charged Higg's fermion. The recently discovered neutral boson of rest energy 120 to 160 MeV seems to be composed of a Higg's charged boson and a W boson. Its obtained mass is 126 GeV. Concepts of modified SUSY and Strong nuclear gravity can be considered for main stream research.

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1 To implement the gravitational constant in nuclear physics

As the culmination of his life's work, Einstein wished to see a unification of gravity and electromagnetism as aspects of one single force. In modern language he wished to unite electric charge with the gravitational charge (mass) into one single entity. Further, having shown that 'mass'- the 'gravitational charge' was connected with space-time curvature, he hoped that the electric charge would likewise be so connected with some other geometrical property of space-time structure. For Einstein the existence, the mass, the charge of the electron and the proton as the only elementary particles recognized back in 1920's were arbitrary features. One of the main goals of a unified theory should explain the existence and calculate the properties of matter. The fundamental question to be answered is: is mass an inherent property of any elementary particle? Electroweak physics assumes that the Higg's boson is the 'mass generator' of all the massive elementary particles. Physicists say, if the strength of the strong interaction is unity, with reference to the strong interaction, the strength of gravitation is 10^{-39}.

The idea of 'strong gravity' originally referred specifically to the mathematical approach of Abdus Salam for the unification of gravity and quantum chromo-dynamics, but is now often used for any particle level gravity approach. For defining the 'strong gravitational constant' a large scaling factor is required. If the scaling factor is a known one, then to some extent, its historical data and physics background makes and brings the unification concepts into one stream. Compared to the current research - it may be in the main streamline or secondary streamline - it can be decided by future thoughts and experiments. Whether it may be 'real' or 'effective', if it exists as a 'single constant' its physical significance can be understood. The 'nuclear size' can be fitted by the 'nuclear Schwarzschild radius'. The 'nucleus' can be considered as 'a strong nuclear black hole'. Nuclear binding energy

constants can be generated directly. Proton-neutron stability can be studied. The origin of 'strong coupling constant' and 'Fermi's weak coupling constant' can be understood. Charged lepton masses can be fitted. The mystery of Higg's physics and electroweak physics can be understood.

It can be supposed that the elementary particles' construction is much more fundamental than the black hole's construction. If one wishes to unify electroweak, strong and gravitational interactions it is a must to implement the classical gravitational constant G_C in the sub atomic physics. By any reason if one implements the Planck scale in elementary particle physics and nuclear physics automatically G_C comes into subatomic physics. Then a large 'arbitrary number' has to be considered as a proportionality constant. With this large arbitrary number it is possible to understand the mystery of the strong interaction and strength of gravitation. The basic and important problem is : How to select the 'arbitrary number' ? For this purpose the 'mole' concept can be considered a fundamental tool. The combination of Avogadro's number and the classical gravitational constant generates the 'effective' 'strong gravitational constant'. This can be considered as the beginning of 'strong nuclear gravity'.

In particle physics, super symmetry (often abbreviated SUSY) is a symmetry that relates elementary particles of one spin to other particles that differ by half a unit of spin and are known as super partners. In a theory with unbroken super symmetry, for every type of boson there exists a corresponding type of fermion with the same mass and internal quantum numbers, and vice-versa. Considering the concepts of 'strong nuclear gravity' and 'SUSY' in this paper an attempt is made to understand the mystery of Higgs and electroweak physics.

1.1 Interesting ideas and observations

The accuracy of the measured value of the (classical) gravitational constant G_C has increased only modestly since the original Cavendish experiment. G_C is quite difficult to measure, as gravity is much weaker than the other fundamental forces, and an experimental apparatus cannot be separated from the gravitational influence of other bodies. Furthermore, gravity has no established relation to the other fundamental forces, so it does not appear

possible to calculate it indirectly from other constants that can be measured more accurately, as is done in some other areas of physics. Published values of G_C have varied rather broadly, and some recent measurements of high precision are, in fact, mutually exclusive. In 2007 P.J. Mohr and B.N. Taylor [1] recommended a value of $G_C \cong 6.6742867 \times 10^{-11}$ $m^3Kg^{-1}sec^{-2}$. Based on the new interferometry techniques A. Bertoldi et al [2] measured $G_C = 6.693 \times 10^{-11}$ $m^3Kg^{-1}sec^{-2}$, where the standard mean error is $\pm 0.027 \times 10^{-11}$ $m^3Kg^{-1}sec^{-2}$ and the systematic error is $\pm 0.021 \times 10^{-11}$ $m^3Kg^{-1}sec^{-2}$. In implementing and estimating its value from atomic and nuclear physical constants, authors observed the following interesting ideas:

1. For any one elementary particle of mass m_0, the magnitude of the gravitational constant is G_C only. As the number of particles increases to Avogadro's number (N), the magnitude of the gravitational constant approaches $N.G_C$. The mass of the system approaches $N.m_0$. This idea leads to "nuclear strong gravity" in the following way:

2. Based on strong gravity, similar to the 'Schwarzschild radius', the size of the system is $R_N \cong \frac{2(NG_C)(Nm_0)}{c^2}$. The volume of one particle is equal to the total volume divided by Avogadro's number $= \frac{4\pi}{3N}R_N^3$.

3. As suggested by Seshavatharam [3] similar to the classical force limit $\frac{c^4}{G_C}$, the force required to bind N particles is $\frac{c^4}{(NG_C)}$. The force required to bind one particle is $\frac{1}{N} \cdot \frac{c^4}{(NG_C)} \cong \frac{c^4}{N^2G_C}$. The rest energy of an electron is very close to $m_ec^2 \cong 8\pi\sqrt{\hbar c \cdot \frac{c^4}{N^2G_C}} \cong \frac{8\pi}{\sqrt{\alpha}}\sqrt{\frac{e^2}{4\pi\epsilon_0} \cdot \frac{c^4}{N^2G_C}} \cong 0.50952547$ MeV.

4. Very interestingly it is noticed that, $\ln\sqrt{\frac{e^2}{4\pi\epsilon_0 G_C m_p^2}} \cong \sqrt{\frac{m_p}{m_e} - \ln(N^2)}$. Here $\frac{e^2}{4\pi\epsilon_0 G_C m_p^2}$ is the ratio of the electromagnetic and gravitational forces of a proton. m_p and m_e are the rest masses of a proton and an electron respectively. This is a very accurate, simple and strange expression. The obtained $G_C \cong 6.666270142 \times 10^{-11}$ $m^3Kg^{-1}sec^{-2}$. Here the number of inputs is only 4. The famous $\hbar$ or α do not appear in this expression! This is a great surprise. Please note that the gravitational constant is the least-well measured fundamental constant in nature.

5. If the nuclear charge radius or characteristic size of a nucleus is $R_0 \approx 1.20$ fermi, its volume $V_0 \cong \frac{4\pi}{3}R_0^3$ and the total volume of N nucleons is $V_N \cong N \cdot \frac{4\pi}{3}R_0^3$. Thus, the size of N nucleons is $R_N \cong N^{\frac{1}{3}}R_0 \cong \frac{2(NG_C)(Nm_0)}{c^2}$. Then rest energy of nucleon comes out to be $m_0c^2 \approx 105\ MeV$. This does not match the rest energy of a nucleon but does match the geometric mean of the nucleon rest energy and its pairing energy constant, 12 MeV. If α_s is the strong coupling constant, it is noticed that $\frac{1}{\alpha_s}m_0c^2 \approx 939$ MeV and $\alpha_s \cdot m_0c^2 \approx 12$ MeV. Moreover it is noticed that $m_0c^2 \approx 105$ MeV is close to "half of the QCD energy scale" ≈ 217 MeV. It is also noticed that $\ln\sqrt{\frac{c^4}{N^2G_C} \div \frac{e^2}{4\pi\varepsilon_o R_N^2}} \cong \frac{1}{\sqrt{\alpha}}$.

6. Considering m_0 as a characteristic strong interaction energy constant, it is noticed that $\sqrt{e^{\frac{m_0}{m_e}} \times \frac{G_C m_e^2}{c}} \cong e^{\frac{m_0}{2m_e}} \times \frac{G_C m_e^2}{c} \cong \hbar$. Here m_e is the rest mass of an electron. If so, it can be represented as $\frac{m_0}{m_e} \cong \ln\left(\frac{\hbar c}{G_C m_e^2}\right)^2 \cong 206.1113643$ and $m_0c^2 \cong 105.3226825$ MeV.

7. J. Erler and P. Langacker [4] presented their views on the strong coupling constant in detail. Considering the strong interaction, the inverse of the 'strong coupling constant ' is $\frac{1}{\alpha_s} \approx \ln\sqrt{\frac{2(G_C)(Nm_0)}{c^2} \div \frac{\hbar}{(Nm_0)c}} \cong 8.902049259 \cong \frac{1}{0.112333685} \cong X_S$. The interpretation is that $\frac{2(G_C)(Nm_0)}{c^2}$ is the black hole radius of (Nm_0) in free space and $\frac{\hbar}{(Nm_0)c}$ is the Compton length of (Nm_0). The natural logarithm of the square root of the ratio of these two lengths represents the inverse of the 'strong coupling constant'. But this is a very hypothetical concept.

8. Another observation is $R_0 \cong \left(\frac{m_e}{m_0}\right)^3 \cdot \frac{NG_C\sqrt{(Nm_p)(Nm_e)}}{c^2} \cong 1.2$ fermi where m_p is the rest mass of a proton. Here R_0 seems to be the minimum distance between (Nm_p) and (Nm_e).

9. It also noticed that $R_0 \cong \frac{1}{N^2} \cdot \left(\frac{\hbar c}{Gm_e^2}\right)^2 \cdot \frac{2G_C m_e}{c^2} \cong \frac{e^{\frac{m_0}{m_e}}}{N^2} \cdot \frac{2G_C m_e}{c^2}$. It can be expressed as $\hbar \cong m_e\sqrt{\frac{(NG_C)(Nm_e)R_0}{2}}$ or $\frac{\hbar}{m_e} \cong \sqrt{\frac{(NG_C)(Nm_e)R_0}{2}}$. Whether it is real or imaginary, this is a very strange and interesting relation.

10. Considering the rest mass of electron, its gravitational mass generator can be defined as $X_E \cong \sqrt{\left(\frac{4\pi\varepsilon_0}{e^2} \cdot \left(\frac{N^2 G_C}{c^4}\right)\right) \cdot m_e^2 c^4} \cong 295.0606338$. Using this number, tau and muon masses can be fitted accurately. The product (αX_E) can be considered as the ratio of down and up quark masses and can also be considered as the 'inverse' of the weak coupling angle. The inverse of the 'strong coupling constant' can also be expressed as $\frac{1}{\alpha_s} \cong \ln\left(X_E^2\sqrt{\alpha}\right) \cong 8.91424 \cong X_S$. It is also noticed that in the hydrogen atom $\frac{a_0}{R_0} \cong \frac{X_E^2}{2}$ where a_0 is the Bohr radius of an electron.

11. The total energy of an electron in a hydrogen atom is $\left(\frac{\alpha^2}{2}\right) m_e c^2 \cong F_W \left(\frac{\hbar}{m_0 c}\right)^{-3}$ where F_W is the Fermi's famous weak coupling constant. $F_W \cong \left(\frac{\alpha^2}{2}\right) m_e c^2 \left(\frac{\hbar}{m_0 c}\right)^3 \cong \left(\frac{m_e}{m_0}\right) \left(\frac{e^2}{4\pi\varepsilon_o m_0 c^2}\right)^2 \left(\frac{\hbar c}{2}\right) \approx \frac{1}{3}\left(\frac{e^2}{4\pi\varepsilon_o m_0 c^2}\right)^3 m_0 c^2$.

2 To fit the gravitational constant with more than 6 atomic constants

For any experimental physicist it is a must to measure or estimate the magnitude of the classical gravitational constant in nuclear physics. It is well established that, in β decay, the neutron emits an electron and transforms to a proton. Thus, the nuclear charge changes and the nucleus gains stability. Based on the semi empirical mass formula, W. D. Myers et al [5] estimated the atomic masses. From the semi empirical mass formula, it is established that,

$$Z \cong \frac{A}{2 + (E_c/2E_a)\, A^{2/3}}. \tag{1}$$

where Z = the number of protons of the stable nucleus and A =the number of nucleons in the stable nucleus. E_a and E_c are the asymmetry and coulombic energy constants. Semi empirically it is noticed that,

$$A_S \cong 2Z + \frac{Z^2}{S_f} \cong 2Z + \frac{Z^2}{157.069} \tag{2}$$

Here S_f is a new number and can be called the nuclear stability factor and A_S is the stable mass number. P. Roy Chowdhury et al [6] proposed a similar relation. With reference to the ratio of neutron and electron rest masses, S_f can be expressed as

$$S_f \cong \sqrt{\alpha} \cdot \frac{m_n}{m_e} \cong 157.0687113 \tag{3}$$

Here α is the fine structure ratio. If Z= 21, A_S = 44.8, Z= 29, A_S = 63.35, Z=47, A_S = 108.06, Z=79, A_S = 197.73 and Z=92, A_S = 237.88. By considering A as the fundamental input its corresponding stable Z = Z_S takes the following form.

$$Z_S \cong \left[\sqrt{\frac{A}{157.069} + 1} - 1\right] 157.069 \tag{4}$$

W. D. Myers and W.J. Swiatecki [7] discussed the Green's stability line. Thus Green's stability formula in terms of Z takes the following form.

$$\frac{0.4A^2}{A+200} \cong A_S - 2Z \cong \frac{Z^2}{S_f}. \tag{5}$$

Surprisingly it is noticed that this number S_f plays a crucial role in fitting the nucleons rest mass. Another interesting observation is that

$$(m_n - m_p)\, c^2 \cong \ln\left(\sqrt{S_f}\right) m_e c^2 \cong 1.29198 \text{ MeV} \tag{6}$$

Here m_n, m_p and m_e are the rest masses of neutron, proton and electron respectively. Semi empirically it is noticed that

$$\frac{E_c}{2E_a} \cdot \frac{e^{S_f}}{N} \cong \frac{e^2}{4\pi\varepsilon_0 G_C m_e^2} \tag{7}$$

The electron rest mass can be expressed as

$$m_e \cong \sqrt{\frac{2E_a}{E_c} \cdot \frac{N}{e^{S_f}}} \cdot \sqrt{\frac{e^2}{4\pi\varepsilon_0 G_C}} \tag{8}$$

Here N is Avogadro's number. Please refer to the P.J. Mohr and B.N. Taylor [1] recommended value. $\frac{e^2}{4\pi\varepsilon_0 G_C m_e^2}$ is the ratio of the electromagnetic and gravitational forces of an electron. In this proposal the important questions are: What is the role of Avogadro's number in β decay ? and How does one interpret the expression $\sqrt{\frac{e^2}{4\pi\varepsilon_0 G_C}}$? This is a multipurpose expression. Either the value of Avogadro's number or the value of the gravitational constant can be fitted. From the semi empirical mass formula if E_a= 23.21 MeV and E_c= 0.71 MeV,

$$G_C \cong \frac{2E_a}{E_c} \cdot \frac{N}{e^{S_f}} \cdot \frac{e^2}{4\pi\varepsilon_0 m_e^2} \cong 6.6866323 \times 10^{-11} \ \mathrm{m^3 Kg^{-1} sec^{-2}} \qquad (9)$$

The obtained value is very close to the recommended value of the classical gravitational constant $6.6742867 \times 10^{-11}$ $\mathrm{m^3 Kg^{-1} sec^{-2}}$. Since all other atomic constants are well measured, the accuracy of G_C only depends upon the E_a and E_c of the semi empirical mass formula.

3 The beginning of strong nuclear gravity

Multiplying and dividing the RHS of equation (8) by N

$$m_e c^2 \cong \sqrt{\frac{2E_a}{E_c} \cdot \frac{N^3}{e^{S_f}}} \cdot \sqrt{\frac{e^2}{4\pi\varepsilon_0} \cdot \frac{c^4}{N^2 G_C}} \cong X_E \cdot \sqrt{\frac{e^2}{4\pi\varepsilon_0} \cdot \frac{c^4}{N^2 G_C}} \qquad (10)$$

where $X_E \cong \sqrt{\frac{2E_a}{E_c} \cdot \frac{N^3}{e^{S_f}}}$ can be called the 'gravitational mass generator' of an electron. $\frac{c^4}{G_C}$ can be considered the 'limit of classical force'. In a grand unified scheme it can be suggested that

$$\frac{\text{Effective atomic gravitational constant}}{\text{Classical gravitational constant}} \cong \frac{G_A}{G_C} \cong \frac{N^2 G_C}{G_C} \cong N^2 \qquad (11)$$

$$X_E \cong \sqrt{\frac{4\pi\varepsilon_0 G_A m_e^2}{e^2}} \cong 295.0606338 \qquad (12)$$

$$\sqrt{\frac{e^2}{4\pi\varepsilon_0 G_A}} \cong 3.087291597 \times 10^{-33} \ \mathrm{Kg} \qquad (13)$$

$$\frac{c^4}{G_A} \cong 3.337152088 \times 10^{-4} \text{ newton} \tag{14}$$

$$\sqrt{\frac{e^2c^4}{4\pi\varepsilon_0 G_A}} \cong \sqrt{\frac{e^2}{4\pi\varepsilon_0} \cdot \frac{c^4}{G_A}} \cong 1.731843735 \times 10^{-3} \text{ MeV} \tag{15}$$

with $\frac{c^4}{G_A}$ defined as the 'characteristic weak force magnitude' and $\sqrt{\frac{e^2}{4\pi\varepsilon_0} \cdot \frac{c^4}{G_A}}$ defined as the 'characteristic weak energy constant'. The most important observation or definition is that the ratio of 'classical force limit' and 'weak force magnitude' is N^2. These definitions have multiple applications in nuclear and particle physics. At this moment a modern physicist cannot admit these definitions. Their view is that this large value of G_C cannot be incorporated in GTR or existing physics. J.V.Narlikar [8] presented the various views on the gravitational constant. This is absolutely true. Absolute lab measurements of G_C have been made on scales of about 1 cm to 1 meter only. For any experimental physicist it is a must to measure the magnitude of G_C in nuclear physics. Without measuring its value, how can one say that the same value of G_C operates in the atomic or nuclear space time curvature? Here the very important question to be answered is: which is more fundamental G_C or N^2G_C?

4 Basic ideas in strong nuclear gravity

In our previous works [9-17] and based on strong nuclear gravity ([18]-[31]), it is suggested that,

1. The magnitude of the weak nuclear force is $F_W \cong \frac{c^4}{G_A} \cong 3.337152088 \times 10^{-4}$ newton. The nuclear strong force and weak force magnitudes correlated with the relation $\sqrt{\frac{F_S}{F_W}} \cong 2\pi \ln\left(\frac{G_A}{G_C}\right) \cong 2\pi \ln\left(N^2\right)$. Thus $F_S \cong 157.9944058$ newton.

2. The fine structure ratio is $\frac{1}{\alpha} \cong \frac{1}{2}\sqrt{X_E^2 - \left[\ln\left(N^2\right)\right]^2} \cong 136.9930484 \cong \frac{1}{2}\sqrt{X_E^2 - \left(\frac{F_S}{4\pi^2 F_W}\right)} \cong 137.036.$

3. The characteristic nuclear size is $R_0 \cong \sqrt{\frac{e^2}{4\pi\varepsilon_0 F_S}} \cong 1.208398568$ fermi.

4. The nuclear weak energy constant is $E_W \cong \sqrt{\frac{e^2 F_W}{4\pi\varepsilon_0}} \cong 1.731843735 \times 10^{-3}$ MeV.

5. The muon and tau rest energy $= (mc^2)_n \cong \left[X_E^3 + (n^2 X_E)^n \sqrt{N}\right]^{\frac{1}{3}} E_W$ where $n = 1, 2$. At n $= 1$, $(mc^2) \cong 105.95$ MeV, n $= 2$, $(mc^2) \cong 1777.4$ MeV.

6. The proton-nucleon nuclear stability factor is $S_f \cong X_E - \frac{1}{\alpha} - 1 \cong 157.0246441$. The proton and nucleon stability relation can be expressed as, stable mass number $= A_S \cong 2Z + \frac{Z^2}{S_f}$ where Z is the proton number.

7. At n $=$ 1 and n $=$ 2, with reference to the electron rest mass , neutron and proton rest masses take the following common expression, $(mc^2)_n \cong \frac{S_f}{\sqrt{\alpha}} m_e c^2 - x\left(2^x + \frac{E_c}{2E_a}\right) m_e c^2$ where $x = (-1)^n$, E_c and E_a are the coulombic and asymmetry energy constants of the semi empirical mass formula.

8. The nuclear strong energy constant is $E_S \cong \sqrt{\frac{e^2 F_S}{4\pi\varepsilon_0}} \cong 1.191630$ MeV and the nuclear coulombic energy constant is $E_c \cong \frac{3}{5} E_S \cong 0.715$ MeV.

9. The proton rest energy is $m_p c^2 \cong \left(\frac{F_S}{F_W} + X_E^2 - \frac{1}{\alpha^2}\right) E_W \cong 938.18$ MeV. The neutron, proton rest energy difference is $(m_n - m_p)\, c^2 \cong \sqrt{\frac{F_S}{F_W} + X_E^2} \cdot E_W \cong 1.29657348$ MeV.

10. The electroweak energy scale is $\varepsilon_W \cong \frac{F_S}{F_W} \times m_e c^2 \cong \frac{F_S}{F_W} \times 0.511 \cong 241927.75$ MeV. This is a very simple confirmation for the proposed definitions of F_S and F_W.

11. The weak coupling angle is $\sin\theta_W \cong \frac{1}{\alpha X_E} \cong 0.464433353 \cong \frac{\text{Up quark mass}}{\text{Down quark mass}}$.

12. The relation between the electron rest mass and the up quark rest mass is $\frac{Uc^2}{m_e c^2} \cong \left[\frac{G_A m_e^2}{\hbar c}\right]^{\frac{1}{3}} \cong 8.596650881 \cong e^{\alpha X_E}$. The relation between the up quark and down quark rest masses is $\frac{Dc^2}{Uc^2} \cong \ln\left[\frac{Uc^2}{m_e c^2}\right] \cong 2.151372695 \cong \alpha X_E$.

13. The magnetic moment of an electron is $\mu_B \cong \frac{ec}{2}\sqrt{\frac{e^2}{4\pi\varepsilon_0 F_W}}\sin\theta_W$ and the magnetic moment of a nucleon is $\mu_n \cong \frac{ec}{2}\sqrt{\frac{e^2}{4\pi\varepsilon_0 F_S}}\sin\theta_W \cong \frac{ecR_0}{2}\sin\theta_W$ where R_0 is the nuclear charge radius or nucleon size.

14. The electron rest energy is $m_e c^2 \cong 2\sin^2\theta_W \times \sqrt{\frac{e^2 F_S}{4\pi\varepsilon_0}}$ and the total energy of an electron in a hydrogen atom can be expressed as $\frac{e^2}{8\pi\varepsilon_0 a_0} \cong \frac{1}{2}\alpha^2 m_e c^2 \cong \frac{1}{X_E^2} \times \sqrt{\frac{e^2 F_S}{4\pi\varepsilon_0}} \cong \frac{1}{X_E^2} \times \frac{e^2}{4\pi\varepsilon_0 R_0}$. Here a_0 is the Bohr radius and R_0 is the characteristic nuclear size. Thus it is noticed that $\hbar \cong \sqrt{\frac{G_A m_e^3 R_0}{2}}$ and $\frac{a_0}{R_0} \cong \frac{X_E^2}{2}$.

15. $X_S \cong \ln\left(X_E^2\sqrt{\alpha}\right) \cong 8.914239916 \cong \frac{1}{\alpha_s}$ can be considered as the 'inverse of the strong coupling constant'.

16. With reference to the proton rest energy, the semi empirical mass formula coulombic energy constant is $E_c \cong \frac{\alpha}{X_S} \cdot m_p c^2 \cong \alpha \cdot \alpha_s \cdot m_p c^2 \cong$ 0.7681 MeV.

17. The pairing energy constant is close to $E_p \cong \frac{m_p c^2 + m_n c^2}{S_f} \cong 11.959$ MeV and the asymmetry energy constant can be expressed as $E_a \cong 2E_p \cong$ 23.918 MeV.

18. Volume and surface energy constants and asymmetric and pairing energy constants can be related by $E_a - E_v \cong E_s - E_p \cong (X_S + 1)\,E_c \cong$ 7.615 MeV. $E_v + E_s \cong E_a + E_p \cong 3E_p$. Thus $E_v \cong 16.303$ MeV and $E_s \cong 19.574$ MeV

4.1 Basic ideas in 'modified' super symmetry

1. There exists a strongly interacting confined fermion of rest energy $M_{Sf}c^2 \cong 105.3226825$ MeV. Its boson rest energy can be expressed as $M_{Sb}c^2 \cong \frac{M_{Sf}c^2}{\Psi} \cong 46.6$ MeV where $\Psi \approx 2.26$ is the fermion and boson mass ratio. In particle physics these mass units play a very interesting role.

2. It is assumed that $m_e \cong \alpha.\sqrt{M_{Sf}.M_{Sb}} \cong \alpha.\sqrt{M_{Sf}.\left(\frac{M_{Sf}}{\Psi}\right)}$. In this way, the value of Ψ is fitted. If m_f and m_b are the rest masses of fermion

and boson, $m_b \cong \frac{m_f}{\Psi}$. The interesting point is that $\left(1-\frac{1}{\Psi}\right) m_f$ acts as the effective fermion.

3. In the previous papers it is also suggested that 'there exists another strongly interacting fermion at $M_{Gf}c^2 \cong [X_S(1-\sin\theta_W)]^3 . M_{Sf}c^2 \cong$ 11460.81321 MeV' and boson at $M_{Gb}c^2 \cong \frac{M_{Gf}c^2}{\Psi} \cong$ 5066.18335 MeV. In this paper with reference to the Higg's charged fermion, it is noticed that the purpose of 11460 MeV can be obtained with the Higg's charged fermion and its boson. Hence this assumption can be discarded.

4. From the above results, on simplification Fermi's weak coupling constant takes the simple form $F_W \cong \frac{1}{2}\left(\frac{e^2}{4\pi\varepsilon_0 M_{sf}c^2}\right)^3 \sqrt{M_{Sf}M_{Sb}} \cdot c^2 \cong$ $1.43358632 \times 10^{-62}\ Jm^3$.

5. The characteristic nuclear fermion is $X_S . M_{Sf}c^2 \cong 938.8716604$ MeV and its corresponding nuclear boson is $M_{Sb}c^2 \cong \frac{X_S M_{Sf}c^2}{\Psi} \cong \frac{938.8716604}{\Psi} \cong$ 415.0225543 MeV. This boson is the mother of presently believed strange mesons like 493, 548, 1020 MeV etc.

6. Fine rotational levels of any ground state energy $m_x c^2$ are, if n =1,2,3, $(mc^2)_I \cong [n(n+1)]^{\frac{1}{4}} m_x c^2 \cong [I]^{\frac{1}{4}} m_x c^2$ and $(mc^2)_{I/2} \cong \left[\frac{n(n+1)}{2}\right]^{\frac{1}{4}} m_x c^2$ $\cong \left[\frac{I}{2}\right]^{\frac{1}{4}} m_x c^2$. Super fine rotational levels are $(mc^2)_I \cong [n(n+1)]^{\frac{1}{12}} m_x c^2$ $\cong [I]^{\frac{1}{12}} m_x c^2$ and $(mc^2)_{I/2} \cong \left[\frac{n(n+1)}{2}\right]^{\frac{1}{12}} m_x c^2 \cong \left[\frac{I}{2}\right]^{\frac{1}{12}} m_x c^2$.

7. For any massive particle, $\frac{\text{mass}}{X_E} \cong \frac{\text{mass}}{295.0606338}$ can be called its electromagnetic mass. Then the fine structure ratio takes the form, $\alpha^2 \cong \frac{e^2}{4\pi\varepsilon_0 G_A}\left(\frac{M_{Sf}}{X_E}\right)^{-1}\left(\frac{M_{Sb}}{X_E}\right)^{-1}$.

8. There exist nature friendly 'integral charge quark fermions' and 'integral charge quark bosons'.

9. There exist integral charged massive quark fermi-gluons and integral charged massive quark boso-gluons. (Fermi-gluon means massive gluons having fermion behaviour and boso-gluon means massive gluons

having boson behaviour. The quark femi-gluon can be called the 'quark baryon' and the quark boso-gluon can be called the 'quark meson').

10. The quark fermi-gluon or quark baryon rest energy is equal to $Q_A c^2 \cong \left[M_{Gf}^2 \times Q_f\right]^{\frac{1}{3}} c^2$ and the quark boso-gluon or quark meson rest energy is equal to $Q_B c^2 \cong \left[M_{Gb}^2 \times Q_b\right]^{\frac{1}{3}} c^2$ where Q_f and Q_b are the rest masses of quark fermion and quark boson respectively.

11. $Q_{ef} \cong Q_f - Q_b \cong \left(1 - \frac{1}{\Psi}\right) Q_f$ acts as the effective quark fermion. The effective quark baryon mass is equal to $Q_E c^2 \cong \left[M_{Gf}^2 \times Q_{ef}\right]^{\frac{1}{3}} c^2$. These effective quark baryons play a vital role in fitting the unstable baryon masses. The quark meson masses play a vital role in fitting the unstable meson masses.

12. '**Integral charge light quark bosons**' in one or two numbers couple with the ground or excited **effective quark baryons** and generate **doublets and triplets**. This is just like the 'absorption of photons by the electron'.

5 Top quark fermion and its SUSY boson

In the previous papers, the authors suggested that up, strange and bottom quarks are in geometric series. Similarly down, charm and top quarks are in another geometric series. Please see the following table 1 for the obtained quark 'fermion' and 'boson' masses. The obtained quark fermion masses can be compared with the current estimates proposed in H. Terazawa's [32] unified composite model. The observed baryon and meson charge-mass spectrum can be generated from these mass units. Please refer to Seshavatharam and Lakshminarayana's published works [9-11]. Proposed up and down fermion masses are

$$U_f c^2 \cong e^{\alpha X_E} \times m_e c^2 \cong 4.400744 \text{ MeV} \tag{16}$$

$$D_f c^2 \cong \alpha X_E \times U_f c^2 \cong 9.47551264 \text{ MeV} \tag{17}$$

Quark	Q_f = **Quark fermion mass** in MeV	Q_b = **Quark boson mass** in MeV
Up	4.401	1.945
Down	9.4755	4.189
Strange	152.5427	67.43
Charm	1313.796	580.756
Bottom	5287.579	2337.34
Top	**182160.18**	**80522.81**

Table 1: Fitting of quark fermion and quark boson masses.

Here, $m_e c^2$= rest energy of electron, α= fine structure ratio , X_E = proposed lepton mass generator. It is very interesting to note that

$$\frac{\text{Down fermion mass}}{\text{Up fermion mass}} \cong \frac{D_f}{U_f} \cong \alpha X_E \cong \frac{1}{\sin(\theta_W)} \tag{18}$$

In this way $\sin\theta_W$ can be related with up and down quark mass ratio. The proposed USB geometric ratio is

$$g_U \cong \left[\alpha X_E \frac{\alpha X_E + 1}{\alpha X_E - 1}\right]^2 \cong 34.66294 \tag{19}$$

If the DCT series is the second generation series, its geometric ratio is

$$g_D \cong \left[2\alpha X_E \frac{\alpha X_E + 1}{\alpha X_E - 1}\right]^2 \cong 138.651754 \quad \text{and} \tag{20}$$

$$\frac{g_D}{g_U} \cong \frac{\text{DCT geometric ratio}}{\text{USB geometric ratio}} \cong 4. \tag{21}$$

$$\text{Quark boson mass } = Q_b \cong \frac{\text{Quark fermion mass}}{\Psi} \cong \frac{Q_f}{\Psi} \tag{22}$$

The obtained top quark boson rest energy is 80522.81 MeV and is very close to the observed W boson rest energy 80.450± 0.058 GeV and 80.392±0.039 GeV. Please refer to M. Yao et al [33] recommended PDG data. Really this

is a great coincidence and support for the proposed new idea of "fermion-boson" unification scheme. This strongly supports super symmetry with small modifications.

5.1 Up and down baryons and rest energy of nucleons

For up and down quarks their baryon masses are

$$U_A c^2 \cong \left[M_{Gf}^2 \times U_f\right]^{\frac{1}{3}} c^2 \cong 833.0140538 \; MeV \tag{23}$$

$$D_A c^2 \cong \left[M_{Gf}^2 \times D_f\right]^{\frac{1}{3}} c^2 \cong 1075.66704 \; MeV \tag{24}$$

where $M_{Gf}c^2 \cong 11460.81321$MeV and plays a crucial role in fitting the observed baryonic mass spectrum. Its bosonic form $M_{Gb}c^2 \cong 5066.18335$ MeV plays a crucial role in fitting the observed mesonic mass spectrum.

Here nucleon rest masses seems to lie between actual up and down baryon masses. The PDG recommended values are, proton rest energy $= m_p c^2 = 938.272013$ MeV and neutron rest energy $= m_n c^2 = 939.5653462$ MeV. The harmonic mean mass of actual up and down baryons is

$$\left(\frac{2U_A D_A}{U_A + D_A}\right) c^2 \cong 938.9161598 \; MeV \tag{25}$$

This mass coincides with the average mass of a nucleon $= 938.9186795$ MeV accurately! This may be the secret of the origin of the rest mass of a nucleon. The harmonic mean mass of up and down quark fermions is

$$\left(\frac{2U_f D_f}{U_f + D_f}\right) c^2 \cong 6.010166458 \; MeV \tag{26}$$

It is noticed that

$$\left(\frac{U_f}{D_f}\right)^2 \times \left(\frac{2U_f D_f}{U_f + D_f}\right) c^2 \cong 1.296382928 \; MeV \tag{27}$$

This is close to the neutron-proton mass difference. Thus

$$\alpha^2 X_E^2 \left(m_n - m_p\right) c^2 \cong \frac{\left(m_n - m_p\right) c^2}{\sin^2 \theta_W} \cong \left(\frac{2U_f D_f}{U_f + D_f}\right) c^2 \cong 6.0102 \; MeV \tag{28}$$

The obtained neutron rest energy is $m_n c^2 \cong 939.5643513 \; MeV$ and the proton rest energy is $m_p c^2 \cong 938.2679683 \; MeV$.

6 Higgs physics in SUSY and strong nuclear gravity

Higgs physics suggests that there exists a neutral boson. It is responsible for the origin of elementary particles mass generation. Its existence was proposed in 1964. Electro weak physics suggests that the nuclear weak force is mediated by the W and Z bosons. Based on modified super symmetry and strong nuclear gravity in the previous papers, the authors suggested that there exists a Higg's charged boson. A charged Higg's boson pair generates the neutralized Z boson. It is also suggested that W is the super symmetric boson of the top quark fermion. Certainly these concepts lead to a new physics in the electroweak sector. At least one can answer the question: Why is the W boson mass very high? The fundamental question to be answered is: what is the role of the charged Higg's boson or its SUSY fermion in nuclear weak decay? Based on strong nuclear gravity and SUSY, in this paper the authors presented a simple relation among the charged Higg's fermion mass, electron rest mass, nuclear charge radius and neutral Z boson rest mass.

6.1 Beta decay, Higg's charged fermion and its boson

It is well established that, in Beta decay, an electron is instantaneously created from a neutron and this nuclear weak force is mediated by W and Z bosons. If a W boson is really the SUSY partner of the top quark then the role of the W boson in weak decay seems to be nothing. Its role is taken up by the newly proposed Higg's charged boson of rest energy close to 45.6 GeV. Its rest energy is equal to half the rest energy of neutral Z boson. If N is Avogadro's number, in strong nuclear gravity it is confirmed that, the magnitude of the operating or effective gravitational constant G_A in atomic and nuclear physics is N^2 times the classical gravitational constant G_C. The weak force magnitude is $\frac{c^4}{G_A}$. Semi empirically it is noticed that

$$\frac{G_A m_e}{c^2 R_0} \cong \frac{\Psi M_{Hb}}{m_e} \tag{29}$$

Here, M_{Hb} is the rest mass of the charged Higg's boson and ΨM_{Hb} is its fermionic form. $\Psi \cong 2.262218404$ is a unified SUSY fermion and the boson mass ratio. m_e is the rest mass of an electron, R_0 is the nuclear characteristic charge radius and G_A is the effective atomic gravitational constant. This is a very surprising relation. This relation is very similar to the expressions generally expressed in 'strong gravity'. Giving primary importance to the existence of a charged Higg's boson, the electron rest mass takes the following form

$$m_e \cong \sqrt{\frac{(\Psi M_{Hb}c^2)\, R_0}{G_A}} \tag{30}$$

$$m_e c^2 \cong \sqrt{\frac{(\Psi M_{Hb}c^2)\, c^4 R_0}{G_A}} \cong \sqrt{(\Psi M_{Hb}c^2) \cdot \left(\frac{c^4}{G_A} \cdot R_0\right)} \tag{31}$$

Its beauty is that it couples all the important nuclear physical constants. No arbitrary parameters appear in this expression. The only important thing missing is the neutron. From the above relation, the mass of ΨM_{Hb} or M_{Hb} can be obtained.

$$M_{Hf}c^2 \cong \frac{G_A m_e}{c^2 R_o} \cdot m_e c^2 \qquad \text{and} \tag{32}$$

$$M_{Hb}c^2 \cong \frac{M_{Hf}c^2}{\Psi} \cong \frac{1}{\Psi} \cdot \frac{G_A m_e}{c^2 R_o} \cdot m_e c^2 \tag{33}$$

Here accuracy depends on R_0. Based on strong nuclear gravity it is also noticed that

$$\hbar \cong \sqrt{\frac{G_A m_e^3 R_0}{2}} \quad \Rightarrow \quad \frac{G_A m_e}{c^2 R_o} \cong \frac{1}{2} \cdot \left(\frac{G_A m_e^2}{\hbar c}\right)^2 \tag{34}$$

$$M_{Hf}c^2 \cong \frac{1}{2} \cdot \left(\frac{G_A m_e^2}{\hbar c}\right)^2 \cdot m_e c^2 \cong 103125.417\ MeV \tag{35}$$

$$M_{Hb}c^2 \cong \frac{M_{Hf}c^2}{\Psi} \cong \frac{1}{2\Psi} \cdot \left(\frac{G_A m_e^2}{\hbar c}\right)^2 \cdot m_e c^2 \cong 45585.96854\ MeV \tag{36}$$

One very interesting observation here is if $R_0 \cong 1.21$ fermi,

$$\frac{c^4}{G_A} \cdot R_0 \cong 2.52 \text{ eV}. \tag{37}$$

This energy is very close to the rest energy of a 'neutrino'. The authors are working in this new direction. **Now the most important question to be answered is : How does one justify the existence of the proposed "Higg's charged susy particle"?** The answer depends on the other applications of $(m_{Hb}c^2)^{\pm}$ and $(\Psi m_{Hb}c^2)^{\pm}$ in elementary particle physics. A $(m_{Hb}c^2)^{\pm}$ pair generates the neutral Z boson. $(m_{Hb}c^2)^{\pm}$ couples with $(m_W c^2)^{\mp}$ boson to form another heavy neutral boson of rest energy 126 GeV. $(m_{Hb}c^2)^{\pm}$and $(\Psi m_{Hb}c^2)^{\pm}$ play vital roles in quark meson and quark baryon mass generation respectively. In this connection the authors noticed some interesting semi empirical observations. They are

$$\sqrt{\frac{M_{Hf}}{M_{Sf}}} \left(M_{Hf} \cdot m_e^2\right)^{\frac{1}{3}} c^2 \cong 937.9012715 \text{ MeV} \approx m_p c^2 \tag{38}$$

$$\sqrt{\frac{M_{Hf}}{\sqrt{m_p \cdot m_e}}} \cong 68.62710438 \approx \frac{1}{2\alpha} \cong 68.51799985 \tag{39}$$

$$\left(\frac{M_{Hf}}{M_{Sf}} \cdot \frac{m_p}{M_{Sf}} \cdot \frac{m_e}{M_{Sf}}\right)^{\frac{4}{3}} \cong 147.4794891 \approx \frac{X_E}{2} \cong 147.5303169 \tag{40}$$

Here M_{Sf} is the proposed characteristic nuclear fermion and its rest energy is 105.3226825 MeV. m_p is the proton rest mass and m_e is the rest mass of electron.

6.2 Rest mass of neutral Z boson

From the above estimation, the neutral Z boson rest energy can be obtained in the following form,

$$m_Z c^2 \cong \left(m_{Hb}c^2\right)^{\pm} + \left(m_{Hb}c^2\right)^{\mp} \cong 2m_{Hb}c^2 \cong 91171.93707 \text{ MeV} \tag{41}$$

This obtained value can be compared with the experimental rest energy of Z boson = 91187.621 MeV. Please refer to M. Yao et al [33] recommended

PDG data. This is a very accurate and simple fitting. In a grand unified program this type of fitting can be given a chance. It seems the Z boson can decay into 2 charged particles. Not only that, this idea opens a new window in elementary particle physics.

6.3 Recently discovered boson of rest energy 120 to 160 GeV

Close to the predicted mass energy of the Higg's boson, T. Aaltonen et al [34] recently reported a new boson of mass energy 120 to 160 GeV. The surprising thing is that its existence does not match current theoretical predictions. In this critical situation, with the help of strong nuclear gravity and modified super symmetry concepts, the authors made an attempt to understand the origin of this new boson. In our previous papers it is suggested that the W boson is the super symmetric boson of the top quark fermion and the charged Higg's boson pair generates the neutralized Z boson.

It is noticed that Higg's charged boson and top quark boson couple together to form a new 'neutral' boson of rest energy 126.0 GeV. This is a very interesting observation.

$$\left(M_{Hb}c^2\right)^{\pm} + \left(m_W c^2\right)^{\mp} \cong 126.0 \ \ GeV. \tag{42}$$

Similarly, the W boson pair generates a neutral boson of rest energy 161 GeV. The average mass of 126 GeV and 161 GeV is 143.5 GeV. Surprisingly this is very close to the mass limit (144 GeV) set for the newly discovered boson! It clearly indicates that "there exist 2 new neutral bosons" whose average rest energy is close to 144 GeV. Based on these coincidences, the proposed concepts of strong nuclear gravity and super symmetry can be confirmed and further research and analysis can be carried out positively. The authors request the science community to look into this new approach.

6.4 Quark baryon and quark meson masses with SUSY Higg's charged particle

It is noticed that

$$Q_A c^2 \cong \left[M_{Gf}^2 \cdot Q_f\right]^{\frac{1}{3}} c^2 \cong \frac{1}{2\alpha\left(X_E+1\right)} \left[M_{Hf}^2 \cdot Q_f\right]^{\frac{1}{3}} c^2 \tag{43}$$

$$Q_B c^2 \cong \left[M_{Gb}^2 \cdot Q_b\right]^{\frac{1}{3}} c^2 \cong \frac{1}{2\alpha\left(X_E+1\right)} \left[M_{Hb}^2 \cdot Q_b\right]^{\frac{1}{3}} c^2 \tag{44}$$

Surprisingly with reference to the 'present notion' of $\sin^2\theta_W$ it is noticed that

$$\frac{1}{2\alpha\left(X_E+1\right)} \cong sin^2\theta_{\mathrm{W}} \tag{45}$$

$$Q_A c^2 \cong \left[M_{Gf}^2 \cdot Q_f\right]^{\frac{1}{3}} c^2 \cong \sin^2\theta_W \left[M_{Hf}^2 \cdot Q_f\right]^{\frac{1}{3}} c^2 \tag{46}$$

$$Q_B c^2 \cong \left[M_{Gb}^2 \cdot Q_b\right]^{\frac{1}{3}} c^2 \cong \sin^2\theta_W \left[M_{Hb}^2 \cdot Q_b\right]^{\frac{1}{3}} c^2 \tag{47}$$

$$Q_E c^2 \cong \left[M_{Gf}^2 \cdot Q_{ef}\right]^{\frac{1}{3}} c^2 \cong \sin^2\theta_W \left[M_{Hf}^2 \cdot Q_{ef}\right]^{\frac{1}{3}} c^2 \tag{48}$$

With these observations, in strong nuclear gravity, the proposed existence of $M_{Gf}c^2 \cong 11460$ MeV and its bosonic form $M_{Gb}c^2 \cong 5066$ MeV can be omitted.

Conclusion

Until today no atomic model implemented the gravitational constant in atomic or nuclear physics. Then, whatever may be its magnitude, measuring its value from existing atomic principles is impossible. Its value was measured in the lab within a range of 1 cm to 1 meter only whereas the observed nuclear size is 1.2 fermi. Independent of the existing methods of estimation, in this paper an attempt is made to fit the gravitational constant with atomic physical constants. In a grand unification program this type of fitting can be given a chance.

The authors showed many applications of SUSY and strong nuclear gravity. Based on the proposed concepts, the existence of the charged SUSY

Higg's particle can be confirmed. The authors hope that this paper will bring the subject of 'strong nuclear gravity' into main stream research in a grand unification program. The authors are working in this new direction.

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A COMPUTATION OF THE PARTIAL WIDTH $\Gamma_{Z^0 \to e^+e^-}$

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Abstract

The contribution from higher-loop diagrams to the partial width of the decay $Z^0 \to \ell^+\ell^-$ for all charged leptons is estimated, based on experimentally measured total width and mass of the Z^0 resonance. This theoretical value is shown to be approximately equal to the invisible width of the Z^0 resonance.

Keywords: *partial width, Z^0 resonance*

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1. Introduction

The discovery of the Z^0 boson has raised the problem of determining the possible decays of this particle and its implications for supersymmetric extensions of the standard model. With the development of unified theories of all of the elementary particle interactions, it has been found useful to introduce supersymmetry to improve the coincidence of the extrapolations of the gauge couplings and approximately preserve mass ratios in perturbation theory. However, the supersymmetric particles would imply the existence of new reactions, and an analysis of the decays of vector bosons has been used to establish whether these new particles could exist.

The effect of higher-order terms in perturbation theory shall be examined for the processes $e^+e^- \rightarrow Z^0 \rightarrow \ell^+\ell^-$. From a sum of vacuum polarization diagrams over all orders and two-loop graphs, the contribution to the width of the Z^0 resonance from higher orders in the perturbative expansion is estimated. It is found to coincide with the invisible width, which has been bounded in experiments on Z^0 physics. The theoretical basis for this result is provided.

The number of generations in the standard model is verified by adding partial widths of the leptons and hadrons. A comparison with the total width also shall be used to determine lower bounds for the mass levels of any particles belonging to a supermultiplet.

2. Perturbative Diagrams in e^+e^- Scattering and the Invisible Width of the Z^0 Resonance

The Breit-Wigner formula gives the cross-section for the tree-level process, and contains a smoothing term derived from the finite lifetime of the resonance there would be a modification through higher-loop corrections. The perturbation theory is described through the expansion of the S-matrix, and scattering amplitudes will contain the propagator of the Z^0 boson. While computations in standard perturbation theory often do not necessarily contain terms related to the lifetimes of intermediate states, these factors can be included through the addition of an imaginary part to the propagator. By an application of the four-dimensional relativistic formalism to the connection between the formula for the energy shift and the propagator, it can be shown that a propagator for a massive particle described by the wavefunction $\psi(t) = \psi(0)e^{-\frac{\Gamma}{2}t}e^{-i(Et-\vec{p}\cdot\vec{r})}$ is given by

$$\frac{1}{q^2 - m^2 - im\Gamma + \frac{\Gamma^2}{4}} \tag{2.1}$$

where m is the mass and Γ is the width. Since the uncertainty in the half-width [1] is

$$\Delta\left(\frac{\Gamma}{2}\right) = \frac{\left(\frac{\Gamma}{2}\right)^2}{m} \tag{2.2}$$

and $\Delta\Gamma \sim i\Gamma$, as the pole in the imaginary direction in the complex plane represents an uncertainty in the width, $im\Gamma \sim \frac{\Gamma^2}{2}$. With this identification, the propagator (2.1) can be replaced by

$$\frac{1}{q^2 - m^2 - \frac{i}{2}m\Gamma} \tag{2.3}$$

such that the amplitude remains peaked at the particle mass m.

For the reaction $e^+e^- \rightarrow Z^0 \rightarrow e^+e^-$, substitution of the propagator $\frac{g^{\mu\nu}}{q^2 - M_Z^2 - \frac{i}{2}M_Z\Gamma_Z}$ into the amplitude gives

$$\begin{aligned} j_e^{\nu(Z)} \frac{1}{q^2 - M_Z^2 - \frac{i}{2}M_Z\Gamma_Z}\Bigg[1 + \tilde{\Pi}_{\nu\mu}(q)\frac{1}{q^2 - M_Z^2 - \frac{i}{2}M_Z\Gamma_Z} \\ + \tilde{\Pi}_{\nu\mu_1}(q)\frac{1}{q^2 - M_Z^2 - \frac{i}{2}M_Z\Gamma_Z}\tilde{\Pi}_{\mu_1\mu}(q)\frac{1}{q^2 - M_Z^2 - \frac{i}{2}M_Z\Gamma_Z} + \ldots\Bigg] \\ j_e^{\mu(Z)}(q) \end{aligned} \tag{2.4}$$

with

$$j_e^{\mu(Z)}(q) = \frac{|e_0|}{2\ sin\ \theta_W\ cos\ \theta_W}\bar{v}_e(\vec{p}_2)\gamma^\nu[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_1). \tag{2.5}$$

Since $\tilde{\Pi}(q)$ is replaced by the finite part of the vacuum polarization graph, $\tilde{\Pi}^c(q^2)(q^\mu q^\nu - g^{\mu\nu}q^2)$, the renormalized amplitude equals

$$\begin{aligned} &\frac{e_0^2}{4sin^2\theta_W cos^2\theta_W}(\bar{v}_e(\vec{p}_2)\gamma^\nu[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_1))\frac{1}{q^2 - M_Z - \frac{i}{2}m_Z\Gamma_Z} \\ &\qquad \cdot (\bar{v}_e\gamma_\nu(\vec{p}_4)[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_3)) \\ &+ \frac{e_0^2}{4sin^2\theta_W cos^2\theta_W}\Bigg\{\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{4}(\bar{v}_e(\vec{p}_2)\not{q}[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_1)) \\ &\qquad \cdot \frac{(q^2)^{n-2}\tilde{\Pi}^c(q^2)^{n-1}}{(q^2 - M_Z^2 - \frac{i}{2}M_Z\Gamma_Z)^n}(\bar{v}_e(\vec{p}_4)\not{q}[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_3)) \\ &+ \sum_{n=2}^{\infty}\frac{(-1)^n}{4}(\bar{v}_e(\vec{p}_2)\gamma^\nu[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_1))\frac{(q^2)^{n-1}\tilde{\Pi}^c(q^2)^{n-1}}{(q^2 - M_Z^2 - \frac{i}{2}M_Z\Gamma_Z)^n} \\ &\qquad \cdot (\bar{v}(\vec{p}_4)\gamma_\nu[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_3))\Bigg\} \end{aligned} \tag{2.6}$$

where $g_{Vf}^{(0)} = I_{3Lf} - 2Q_f sin^2\theta_W$ and $g_{Af}^{(0)} = I_{3Lf}$ [4], and the symmetry factor of the vacuum polarization graph is included.

The differential cross-section would be

$$
\begin{aligned}
&\frac{1}{64\pi^2 s}\frac{e_0^4}{16 sin^4\theta_W cos^4\theta_W} \\
&\quad\Big\{ Tr((\bar{v}_e(\vec{p}_2)\gamma^\nu[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_1))(\bar{u}_e(\vec{p}_1[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]\gamma_\mu v_e(\vec{p}_2))) \\
&\qquad Tr((\bar{v}_e(\vec{p}_4)\gamma_\nu[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_3))(\bar{u}_e(\vec{p}_3)[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]\gamma_\mu u_e(\vec{p}_4))) \\
&\quad\Big[\frac{1}{q^2 - M_Z^2 + \frac{1}{4}M_Z^2\Gamma_Z^2} + \sum_{n=2}^{\infty}\frac{(-1)^n}{4}\frac{(q^2)^{n-1}\tilde{\Pi}^c(q^2)^{n-1}}{(q^2 - M_Z^2 - \frac{i}{2}M_Z\Gamma_Z)(q^2 - M_Z^2 + \frac{i}{2}M_Z\Gamma_Z)^n} \\
&\quad + \sum_{n=2}^{\infty}\frac{(-1)^n}{4}\frac{(q^2)^{n-1}\tilde{\Pi}^c(q^2)^{n-1}}{(q^2 - M_Z^2 - \frac{i}{2}M_Z\Gamma_Z)^n(q^2 - M_Z^2 + \frac{i}{2}M_Z\Gamma_Z)} \\
&\quad + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{(-1)^{m+n}}{4^2}\frac{(q^2)^{m+n-2}\tilde{\Pi}^c(q^2)^{m+n-2}}{(q^2 - M_Z^2 - \frac{i}{2}M_Z\Gamma_Z)^m(q^2 - M_Z^2 + \frac{i}{2}M_Z\Gamma_Z)^n}\Big] \\
&\qquad + Tr((\bar{v}_e(\vec{p}_2)\not{q}[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_1))(\bar{u}_e(\vec{p}_1[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]\gamma_\mu v_e(\vec{p}_2))) \\
&\qquad Tr((\bar{v}_e(\vec{p}_4)\not{q}[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_3))(\bar{u}_e(\vec{p}_3)[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]\gamma_\mu u_e(\vec{p}_4))) \\
&\quad\Big[\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{4}\frac{(q^2)^{n-2}\tilde{\Pi}^c(q^2)^{n-1}}{(q^2 - M_Z^2 - \frac{i}{2}M_Z\Gamma_Z)(q^2 - M_Z^2 + \frac{i}{2}M_Z\Gamma_Z)^n} \\
&\quad + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{4}\frac{(q^2)^{n-2}\tilde{\Pi}^c(q^2)^{n-1}}{(q^2 - M_Z^2 - \frac{i}{2}M_Z\Gamma_Z)^n(q^2 - M_Z^2 + \frac{i}{2}M_Z\Gamma_Z)} \\
&\quad + 2\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{(-1)^{m+n-1}}{4^2}\frac{(q^2)^{m+n-3}\tilde{\Pi}^c(q^2)^{m+n-2}}{(q^2 - M_Z^2 - \frac{i}{2}M_Z\Gamma_Z)^m(q^2 - M_Z^2 + \frac{i}{2}M_Z\Gamma_Z)^n}\Big] \\
&\qquad + Tr(v_e(\vec{p}_2)\not{q}[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_1))(\bar{u}_e(\vec{p}_1)[g_{Ve}^{(0)} - \gamma_5 g_{Ve}^{(0)}]\not{q}v_e(\vec{p}_2))) \\
&\qquad Tr((\bar{v}_e(\vec{p}_4)\not{q}[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_3))(\bar{u}_e(\vec{p}_3)[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]\not{q}u_e(\vec{p}_4))) \\
&\quad \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{(-1)^{m+n}}{4^2}\frac{(q^2)^{m+n-4}\tilde{\Pi}^c(q^2)^{m+n-2}}{(q^2 - M_Z^2 - \frac{i}{2}M_Z\Gamma_Z)^m(q^2 - M_Z^2 + \frac{i}{2}M_Z\Gamma_Z)^n}\Big\}.
\end{aligned}
\tag{2.7}
$$

It can be verified that

$$
\begin{aligned}
&Tr((\bar{v}_e(\vec{p}_2\gamma^\nu[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_1))(\bar{u}_e(\vec{p}_1)[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]\gamma^\mu v_e(\vec{p}_2))) \\
&\quad Tr((\bar{v}(\vec{p}_4)\gamma_\nu[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_3))(\bar{u}(\vec{p}_3)[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]\gamma_\mu v_e(\vec{p}_4))) \\
&= 32(p_1\cdot p_3 p_2\cdot p_4 + p_1\cdot p_4 p_2\cdot p_3 + 2m_e^4)[g_{Ve}^{(0)2} - g_{Ae}^{(0)2}]^2
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
&Tr((\bar{v}_e(\vec{p}_2)\not{q}[g_{Ve}^{(0)}-\gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_1))(\bar{u}_e(\vec{p}_1[g_{Ve}^{(0)}-\gamma_5 g_{Ae}^{(0)}]\gamma^\mu v_e(\vec{p}_2)))\\
&\quad Tr((\bar{v}_e(\vec{p}_4)\not{q}[g_{Ve}^{(0)}-\gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_3))(\bar{u}_e(\vec{p}_3))(\bar{u}_e(\vec{p}_3)[g_{Ve}^{(0)}-\gamma_5 g_{Ae}^{(0)}]\gamma_\mu v_e(\vec{p}_4)))\\
&=16(p_1\cdot qp_3\cdot qp_2\cdot p_4+p_1\cdot qp_4\cdot qp_2\cdot p_3+p_1\cdot p_4p_2\cdot qp_3\cdot q+p_1\cdot p_3p_2\cdot qp_4\cdot q\\
&\quad -2p_1\cdot qp_2\cdot q(m_e^2+p_3\cdot p_4)-2p_3\cdot qp_4\cdot q(m_3^2+p_1\cdot p_2)\\
&\quad +(m_e^2+p_1\cdot p_2)(m_e^2+p_3\cdot p_4)q^2)[g_{Ve}^{(0)2}-g_{Ae}^{(0)2}]^2
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
&Tr((\bar{v}_e(\vec{p}_2)\not{q}[g_{Ve}^{(0)}-\gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_1))(\bar{u}_e(\vec{p}_1[g_{Ve}^{(0)}-\gamma_5 g_{Ae}^{(0)}]\not{q}v_e(\vec{p}_2)))\\
&\quad Tr((\bar{v}_e(\vec{p}_4)\not{q}[g_{Ve}^{(0)}-\gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_3))(\bar{u}_e(\vec{p}_3))(\bar{u}_e(\vec{p}_3)[g_{Ve}^{(0)}-\gamma_5 g_{Ae}^{(0)}]\not{q}v_e(\vec{p}_4)))\\
&=16(2p_1\cdot qp_2\cdot q-(m_e^2+p_1\cdot p_2)q^2)(2p_3\cdot qp_4\cdot q-(m_e^2+p_3\cdot p_4)q^2)[g_{Ve}^{(0)2}-g_{Ae}^{(0)2}]^2
\end{aligned}
\tag{2.10}
$$

and

$$
\begin{aligned}
&\sum_{n=2}^{\infty}\frac{(-1)^n}{4}\frac{(q^2)^{n-1}\tilde{\Pi}^c(q^2)^{n-1}}{(q^2-M_Z^2-\frac{i}{2}M_Z\Gamma_Z)^n}\\
&\qquad=\frac{1}{4}\frac{\tilde{\Pi}^c(q^2)q^2}{\left(q^2-M_Z^2-\frac{i}{2}M_Z\Gamma_Z\right)\left((1+\tilde{\Pi}^c(q^2))q^2-M_Z^2-\frac{i}{2}M_Z\Gamma_Z\right)}
\end{aligned}
\tag{2.11}
$$

such that the contribution to the differential cross-section from the tree-elevel diagram and vacuum polarization graphs equals

$$
\begin{aligned}
&\frac{1}{64\pi^2 s}\frac{e_0^4}{sin^4\theta_W cos^4\theta_W}[g_{Ve}^{(0)2}-g_{Ae}^{(0)2}]^2\\
&\qquad\Big\{2(p_1\cdot p_3p_2\cdot p_4+p_1\cdot p_4p_2\cdot p_3+2m_e^4)\\
&\qquad\Big[\frac{1}{q^2-M_Z^2-\frac{i}{2}M_Z\Gamma_Z}+\sum_{n=2}^{\infty}\frac{(-1)^n}{4}\frac{(q^2)^{n-1}\tilde{\Pi}^c(q^2)^{n-1}}{[(q^2-M_Z^2)^2+\frac{1}{4}M_Z^2\Gamma_Z^2]}\\
&\qquad\qquad\cdot\frac{2cos\ (n-1)\alpha}{[(q^2-M_Z^2)^2+\frac{1}{4}M_Z^2\Gamma_Z^2]^{\frac{n-1}{2}}}\\
&\qquad\quad+\frac{1}{16}\frac{q^4\tilde{\Pi}^c(q^2)^2}{[q^2-M_Z^2+\frac{1}{4}M_Z^2\Gamma_Z^2][(1+\tilde{\Pi}^c(q^2))q^2+\frac{1}{4}M_Z^2\Gamma_Z^2]}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ (p_1 \cdot qp_3 \cdot qp_2 \cdot p_4 + p_1 \cdot qp_4 \cdot p_2 \cdot p_3 + p_1 \cdot p_4 p_2 \cdot qp_3 \cdot q + p_1 \cdot p_3 p_2 \cdot qp_4 \cdot q \\
&\quad - 2p_1 \cdot qp_2 \cdot q(m_e^2 + p_3 \cdot p_3 \cdot p_4) - 2p_3 \cdot qp_4 \cdot q(m_e^2 + p_1 \cdot p_2) \\
&\quad + (m_e^2 + p_1 \cdot p_2)(m_e^2 + p_3 \cdot p_4)q^2) \\
&\Big[\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{4} \frac{(q^2)^{n-2}\tilde{\Pi}^c(q^2)^{n-1}}{[(q^2 - M_Z^2)^2 + \frac{1}{4}M_Z^2\Gamma_Z^2]} \cdot \frac{2cos\ (n-1)\theta}{[(q^2 - M_Z^2)^2 + \frac{1}{4}M_Z^2\Gamma_Z^2]^{\frac{n-1}{2}}} \\
&\quad - \frac{1}{8}\frac{q^2\tilde{\Pi}^c(q^2)^2}{[q^2 - M_Z^2 + \frac{1}{4}M_Z^2\Gamma_Z^2][(1 + \tilde{\Pi}^c(q^2))q^2 + \frac{1}{4}M_Z^2\Gamma_Z^2]}\Big] \\
&+ (2p_1 \cdot qp_2 \cdot q - (m_e^2 + p_1 \cdot p_2)q^2)(2p_3 \cdot qp_4 \cdot q - (m_e^2 + p_3 \cdot p_4)q^2) \\
&\cdot \frac{1}{16}\frac{\tilde{\Pi}^c(q^2)^2}{[q^2 - M_Z^2 + \frac{1}{4}M_Z^2\Gamma_Z^2][(1 + \tilde{\Pi}^c(q^2))q^2 + \frac{1}{4}M_Z^2\Gamma_Z^2]}\Big]\Big\}
\end{aligned}
\tag{2.12}
$$

where $tan\ \alpha = \frac{M_Z\Gamma_Z}{2(q^2 - M_Z^2)}$.

Although the value of $\tilde{\Pi}^c(q^2)$ has been conjectured after dimensional regularization, it appears that a rigorous value can be found through dispersion relation techniques. This value would determine the radius of convergence of the series (2.12) since the range of the momenta would be limited by the inequality

$$
\frac{(q^2 - M_Z^2)^2 + \frac{1}{4}M_Z^2\Gamma_Z^2}{q^2} \leq \tilde{\Pi}^c(q^2)^2. \tag{2.13}
$$

For other leptonic final states, the corresponding current is substituted into the amplitude, implying that the mass m_e is changed to m_μ, m_τ, m_ν. Because the cross-section for processes with neutrinos and anti-neutrinos in the final state would contain the expression

$$
\begin{aligned}
&Tr((\bar{v}_e(\vec{p}_2)\gamma^\rho[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_1))(\bar{u}_e(\vec{p}_1)[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]\gamma^\mu v_e(\vec{p}_2))) \\
&\quad Tr((\bar{v}_\nu(\vec{p}_4)\gamma_\rho[g_{V\nu}^{(0)} - \gamma_5 g_{A\nu}^{(0)}]u_\nu(\vec{p}_3))(\bar{u}_\nu(\vec{p}_3)[g_{V\nu}^{(0)} - \gamma_5 g_{A\nu}^{(0)}]\gamma_\mu v_\nu(\vec{p}_4))) \\
&\quad = Tr((\not{p}_2 - m_e)\gamma^\rho[g_{V_e}^{(0)} - \gamma_5 g_{Ae}^{(0)}](\not{p}_1 + m_e)[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]\gamma^\mu) \\
&\qquad Tr(\not{p}_4\gamma_\rho[g_{V\nu}^{(0)} - \gamma_5 g_{A\nu}^{(0)}]\not{p}_3[g_{V\nu}^{(0)} - \gamma_5 g_{A\nu}^{(0)}]\gamma_\mu) \\
&\quad = Tr((\not{p}_2 - m_e)\gamma^\rho(\not{p}_1 + m_e)\gamma^\mu) \\
&\qquad Tr(\not{p}_4\gamma_\rho\not{p}_3\gamma_\mu)[g_{V\nu}^{(0)2} - g_{A\nu}^{(0)2}]^2
\end{aligned}
\tag{2.14}
$$

it is not necessary to include these processes in the computation of the width.

The inclusion of loops consisting of other lepton-anti-lepton pairs yields the following expression with $\tilde{\Pi}^c_{e^+e^-}(q^2)$ replaced by $\tilde{\Pi}^c_{e^+e^-}(q^2) + \tilde{\Pi}^c_{\mu^+\mu^-}(q^2) + \tilde{\Pi}^c_{\tau^+\tau^-}(q^2)$
$\tilde{\Pi}^c_{\nu_e\bar{\nu}_e}(q^2) + \tilde{\Pi}^c_{\nu_\mu\bar{\nu}_\mu}(q^2) + \tilde{\Pi}^c_{\nu_\tau\bar{\nu}_\tau}(q^2)$.

The ratio of higher-order terms to the first term in the expansion of the cross-section can be estimated as follows. The tree-level contribution to the differential cross-section equals

$$\frac{1}{64\pi^2 s}\frac{e_0^4}{sin^4\theta_W cos^4\theta_W}\frac{1}{(q^2-M_Z^2)^2+\frac{1}{4}M_Z^2\Gamma^2} \cdot 2(p_1\cdot p_3 p_2\cdot p_4 + p_1\cdot p_4 p_2\cdot p_3 + 2m_e^2)[g_{Ve}^{(0)2} - g_{Ae}^{(0)2}]^2 \tag{2.15}$$

whereas the summation of higher-loop graphs resulting in the third term is

$$\frac{1}{64\pi^2 s}\frac{e_0^4}{sin^4\theta_W cos^4\theta_W}\frac{1}{(q^2-M_Z^2)^2+\frac{1}{4}M_Z^2\Gamma^2}2(p_1\cdot p_3 p_2\cdot p_4 + p_1\cdot p_4 p_2\cdot p_3 + 2m_e^2) \cdot \frac{1}{16}\frac{\tilde{\Pi}^c(q^2)^2 q^4}{[(1+\tilde{\Pi}^c(q^2))q^2 - M_Z^2)^2 + \frac{1}{4}M_Z^2\Gamma_Z^2]}[g_{Ve}^{(0)2} - g_{Ae}^{(0)2}]^2 \tag{2.16}$$

and the ratio is

$$\frac{1}{16}\frac{\tilde{\Pi}^c(q^2)^2 q^4}{[((1+\tilde{\Pi}^c(q^2))q^2 - M_Z^2)^2 + \frac{1}{4}\Gamma_Z^2 M_Z^2]}. \tag{2.17}$$

At $q^2 \approx M_Z^2$, it equals

$$\frac{1}{16}\frac{\tilde{\Pi}^c(M_Z^2)^2}{\tilde{\Pi}^c(M_Z^2)^2 + \frac{1}{4}\frac{\Gamma_Z^2}{M_Z^2}}. \tag{2.18}$$

From dispersion relation techniques, the sum of the values of $\tilde{\Pi}^c(M_Z^2)$ for the five quark pairs equals 0.02758 ± 0.0035 [2]. Since the masses of the quarks and charged leptons are considerably less than that of the Z^0 boson, it is consistent to set $\tilde{\Pi}^c(M_Z^2) \approx (5.516 \pm 0.070) \times 10^{-3}$ for each of the fermion loops. Then, substituting the values $M_Z = (91.1876 \pm 0.0021)\ GeV$ and $\Gamma_Z = (2.4952 \pm 0.0023)\ GeV$ [3][4], the fraction (2.18) nearly equals 8.726×10^{-3}, while

$$\frac{1}{16}\frac{(6\tilde{\Pi}^c(M_Z^2))^2}{(6\tilde{\Pi}^c(M_Z^2)^2 + \frac{1}{4}\frac{\Gamma_Z^2}{M_Z^2}} \approx (5.3378 \pm 0.1788) \times 10^{-2}. \tag{2.19}$$

The mass M_Z and width Γ_Z do not have to be adjusted in this formula [3][5][6] as a result of the location of the pole in the s-plane, because initial choice of these parameters has been used in the derivation of the differential cross section.

The other sum in the first part of the expression has the form

$$\begin{aligned} -\sum_{n=1}^{\infty}(-1)^n x^n cos\ n\alpha &= -Re\sum_{n=1}^{\infty}(-1)^n x^n e^{in\alpha} \\ &= Re\left\{\frac{xe^{i\alpha}}{1+xe^{i\alpha}}\right\} \\ &= \frac{x\ cos\ \alpha + x^2}{1+2x\ cos\ \alpha + x^2}. \end{aligned} \tag{2.20}$$

The sum equals

$$\begin{aligned} &\frac{1}{4}[(q^2-M_Z^2)^2+\frac{1}{4}\Gamma_Z M_Z^2]^{-1}\cdot 2\frac{x\ cos\ \alpha + x^2}{1+2x\ cos\ \alpha + x^2} \\ &x = \frac{q^2\tilde{\Pi}^c(q^2)}{[(q^2-M_Z^2)^2+\frac{1}{4}\Gamma_Z M_Z^2]^{\frac{1}{2}}}. \end{aligned} \tag{2.21}$$

Since

$$\frac{x\ cos\ \alpha + x^2}{1+2x\ cos\ \alpha + x^2} = \frac{q^2\tilde{\Pi}^c(q^2)(q^2-M_Z^2)+q^4\tilde{\Pi}^c(q^2)^2}{(q^2-M_Z^2)^2+(1+2\tilde{\Pi}^c(q^2))q^4+M_Z^2\left(\frac{1}{4}\Gamma_Z^2-q^2\tilde{\Pi}^c(q^2)\right)} \tag{2.22}$$

the ratio of the second term to the first term in the differential cross-section is

$$\frac{\tilde{\Pi}^c(q^2)}{2}\frac{(1+\tilde{\Pi}^c(q^2))q^4-M_Z^2q^2}{(q^2-M_Z^2)^2+\tilde{\Pi}^c(q^2)(2+\tilde{\Pi}^c(q^2))q^4+M_Z^4\left(\frac{1}{4}\Gamma_Z^2-q^2\Pi^c(q^2)\right)}. \tag{2.23}$$

At $q^2 \approx M_Z^2$, it equals

$$\begin{aligned} &\frac{\tilde{\Pi}^c(M_Z^2)}{2}\frac{(1+\tilde{\Pi}^c(M_Z^2))M_Z^4-M_Z^4}{\tilde{\Pi}^c(M_Z^2)(2+\tilde{\Pi}^c(M_Z^2))M_Z^4+M_Z^2\left(\frac{1}{4}\Gamma_Z^2-M_Z^2\tilde{\Pi}^c(M_Z^2)\right)} \\ &\qquad = \frac{\tilde{\Pi}^c(M_Z^2)^2}{2\left[\tilde{\Pi}^c(M_Z^2)(1+\tilde{\Pi}^c(M_Z^2))+\frac{1}{4}\frac{\Gamma_Z^2}{M_Z^2}\right]}. \end{aligned} \tag{2.24}$$

With the values of $\tilde{\Pi}^c(M_Z^2)$, M_Z and Γ_Z,

$$\begin{aligned} \frac{\tilde{\Pi}^c(M_Z^2)^2}{2\left[\tilde{\Pi}^c(M_Z^2)(1+\tilde{\Pi}^c(M_Z^2))+\frac{1}{4}\frac{\Gamma_Z^2}{M_Z^2}\right]} &= (2.6533\pm 0.0747)\times 10^{-3} \\ \frac{(6\tilde{\Pi}^c(M_Z^2))^2}{2\left[6\tilde{\Pi}^c(M_Z^2)(1+6\tilde{\Pi}^c(M_Z^2))+\frac{1}{4}\frac{\Gamma_Z^2}{M_Z^2}\right]} &= (1.5931\pm 0.0451)\times 10^{-2}. \end{aligned} \tag{2.25}$$

While it is possible to use the center-of-mass frame, and deduce that the net coefficients of the second and third group of terms in Eq.(2.12) are proportional to m_e^2, and therefore, negligible in comparison with the first momentum-dependent coefficient, the reduction in the percentage of the cross-section, and therefore the width, is not large enough for the width derived from higher-loop vacuum polarization graphs to be less than the experimental value of the invisible width of the Z^0 resonance [3]. Consequently, factors of $cos\ \theta_i$ shall be introduced for inner products of p_i with q, together with an averaging over angles of incidence in the initial state, additional powers of $cos\ \theta_i$ occurring in an integration over the solid angle. Specifically, $\int_0^\pi cos^2\theta sin\ \theta_i d\theta_i = \frac{2}{3}$, implying a reduction by a factor of 3, while $\int_0^\pi cos^4\theta_i sin\ \theta_i d\theta_i = \frac{2}{5}$, implying a reduction by a factor of 5.

Reviewing the inner products of the momenta in the latter part of the expression for the differential cross-section, there is is approximate cancellation between $p_1 \cdot qp_3 \cdot qp_2 \cdot p_4 + p_1 \cdot qp_4 \cdot qp_2 \cdot p_3 + p_1 \cdot p_4 p_2 \cdot qp_3 \cdot q + p_1 \cdot p_3 p_2 \cdot qp_4 \cdot q$ and $2p_1 \cdot qp_2 \cdot p_3 \cdot p_4 + 2p_3 \cdot qp_4 \cdot qp_1 \cdot p_2$ on average, leaving terms proportional to the masses of the fermions and $p_1 \cdot p_2 p_3 \cdot p_4 q^2$. This term is approximately $\frac{1}{4}$ of the analogous term in the first part of the expression.

In the last part of the expression for the differential cross-section, the terms not containing the fermion masses comprise

$$\begin{aligned}(2p_1 \cdot qp_2 \cdot q - p_1 \cdot p_2 q^2)(2p_3 \cdot qp_4 \cdot q - p_3 \cdot p_4 q^2) \\ = 4p_1 \cdot qp_2 \cdot qp_3 \cdot qp_4 \cdot q - 2p_1 \cdot qp_2 \cdot qp_3 \cdot p_4 q^2 - 2p_3 \cdot qp_4 \cdot qp_1 \cdot p_2 q^2 \\ + p_1 \cdot p_2 p_3 \cdot p_4 q^4\end{aligned} \tag{2.26}$$

yielding a fraction of $\frac{4}{5} - \frac{2}{3} - \frac{2}{3} + 1 = \frac{7}{15}$. The previously computed percentage would be amended to

$$\begin{aligned}\left(1 - \frac{1}{2} + \frac{7}{15}\right) \cdot (5.3378 \pm 0.1788) \times 10^{-2} + \left(1 - \frac{1}{4}\right) \cdot (1.5931 \pm 0.0451) \times 10^{-2} \\ \doteq (6.3547 \pm 0.1761) \times 10^{-2}.\end{aligned} \tag{2.27}$$

These diagrams therefore contribute a width of

$$\Delta\Gamma_{polariz.} = 3((6.3544 \pm 0.1761) \times 10^{-2})((2.4952 \pm 0.0023)\ GeV) \doteq 475.6874 \pm 13.1890\ MeV \tag{2.28}$$

to the Z^0 resonance.

There are three two-loop diagrams which would represent the largest remaining contribution to the cross-section. Two of the diagrams have self-energy subgraphs, while the

third diagram consists of a combination of two vertex subgraphs. The amplitude for the diagram with the self-energy subgraph attached to an internal electron line is

$$\frac{e_0^4}{16\ sin^4\theta_W cos^4\theta_W}(\bar{v}_e(\vec{p}_2)\gamma_\nu[g_{Ve}^{(0)}-\gamma_5 g_{Ae}^0]u_e(\vec{p}_1))(\bar{v}_e(\vec{p}_4)\gamma_\mu[g_{Ve}^{(0)}-\gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_3))$$
$$\frac{1}{(q^2-M_Z^2-\frac{i}{2}\Gamma_Z M_Z)^2}\int\frac{d^4k}{(2\pi)^4}\frac{Tr\,(\gamma^\nu(\not{k}-m_e)\gamma^\mu(\not{k}-\not{q}+m_e))}{[k^2-m_e^2+i\epsilon][(k-q)^2-m_e^2+i\epsilon]}$$
$$\int\frac{d^4k'}{(2\pi)^4}Tr\Big((\not{k}-\not{q}-m_e)\gamma_{\mu_1}[g_{Ve}^{(0)}-\gamma_5 g_{Ae}^{(0)}]\frac{g^{\mu_1\mu_2}}{k'^2-M_Z^2-\frac{i}{2}\Gamma_Z M_Z}$$
$$(\not{q}+\not{k}'-\not{k}+m_e)\frac{1}{\not{q}+\not{k}'-\not{k}+m_e}\gamma_{\mu_2}[g_{Ve}^{(0)}-\gamma_5 g_{Ae}^{(0)}]\Big) \tag{2.29}$$

which yields, after removal of the divergent part of the vacuum polarization subgraph,

$$\frac{e_0^4}{16\ sin^4\theta_W cos^4\theta}(\bar{v}_e(\vec{p}_2)\gamma_\nu[g_{Ve}^{(0)}-\gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_1))(\bar{v}_e(\vec{p}_4)\gamma_\mu[g_{Ve}^{(0)}-\gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_3))$$
$$\frac{1}{(q^2-M_Z^2-\frac{i}{2}\Gamma_Z M_Z)^2}\tilde{\Pi}^{\mu\nu c}(q^2)(q^\mu q^\nu-g^{\mu\nu}q^2)(-4m_e)[g_{Ve}^{(0)2}-g_{Ae}^{(0)2}]$$
$$\int\frac{d^4k'}{(2\pi)^4}\frac{1}{k'^2-M_Z-\frac{i}{2}\Gamma_Z M_Z}. \tag{2.30}$$

Expanding the n-dimensional integral about $n=4$ and using the identity

$$(r-1)!\int\frac{d^nk}{(k^2+A)^r}=\pi^{\frac{1}{2}n}A^{\frac{1}{2}n-r}\Gamma\left(r-\frac{1}{2}n\right) \tag{2.31}$$

the expression

$$-\frac{e_0^4 m_e M_Z^2}{16\ sin^4\theta_W cos^4\theta}(\bar{v}_e(\vec{p}_2)\gamma_\nu[g_{Ve}^{(0)}-\gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_1))(\bar{v}_e(\vec{p}_4)\gamma_\mu[g_{Ve}^{(0)}-\gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_3))$$
$$\frac{[g_{Ve}^{(0)2}-g_{Ae}^{(0)2}]}{(q^2-M_Z^2-\frac{i}{2}\Gamma_Z M_Z)^2}\tilde{\Pi}^{\mu\nu\ c}(q^2)(q^\mu q^\nu-g^{\mu\nu}q^2)(\gamma-1) \tag{2.32}$$

is derived.

While a factor of $-m_e$ was introduced from the sum $\sum_{(s')}v^{(s')}(\vec{k}-\vec{q})\bar{v}^{(s')}(\vec{k}-\vec{q})$ in this integral, a factor of m_e would arise from $\sum_{(s)}u^{(s)}(\vec{k}'+\vec{q}-\vec{k})\bar{u}^{(s)}(\vec{k}'+\vec{q}-\vec{k})$ in the integral for the diagram with the self-energy subgraph attached to the internal positron line, and the finite remainders cancel.

After subtractions, the amplitude for the third diagram equals

$$\frac{e_0^4}{16 sin^4\theta_W cos^4\theta_W} j_e^{\lambda(Z)} \frac{1}{(q^2 - M_Z^2 - \frac{i}{2}\Gamma_Z M_Z)^2} \\ Tr\left(\bar{v} i \frac{\sigma_{\lambda\mu} q^\mu}{2m_e} u \bar{u} i \frac{\sigma_{\tau\rho} q^\rho}{2m_e} v\right) \tilde{\Delta}_2(q^2)^2 j_e^{\tau(Z)}. \tag{2.33}$$

Including this diagram in the expansion of the S-matrix, the cross-term in the product with the tree amplitude would be

$$\frac{1}{64\pi^2 s} \frac{e_0^6}{64\ sin^6\theta_W cos^6\theta_W} \tilde{\Delta}_2(q^2)^2 \\ Tr((\bar{v}_e(\vec{p}_2)\gamma^{\lambda_1}[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_1))(\bar{u}_e(\vec{p}_1)[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]\gamma^{\lambda_2} v_e(\vec{p}_2))) \\ Tr((\bar{v}(\vec{p}_4)\gamma^{\tau_1}[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]u_e(\vec{p}_3))(\bar{u}_e(\vec{p}_3)[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]\gamma^{\tau_2} u_e(\vec{p}_4))) \\ \left\{\frac{1}{q^2 - M_Z^2 + \frac{i}{2}\Gamma_Z M_Z} \frac{1}{(q^2 - M_Z^2 - \frac{i}{2}\Gamma_Z M_Z)^2} \right. \\ \left. + \frac{1}{(q^2 - M_Z^2 + \frac{i}{2}\Gamma_Z M_Z)^2} \frac{1}{q^2 - M_Z^2 - \frac{i}{2}\Gamma_Z M_Z}\right\} \\ \left[-\frac{1}{4m_e^2} Tr(\gamma^\nu \sigma_{\lambda_1\mu} \gamma^\eta \sigma_{\tau_1\rho}) q_\nu q_\eta q^\mu q^\rho + \frac{1}{4} Tr(\sigma_{\lambda_1\mu}\sigma_{\tau_1\rho}) q^\mu q^\rho\right]. \tag{2.34}$$

The following identities may be used.

$$Tr(\sigma_{\lambda\mu}\sigma_{\tau\rho}) = 4(g_{\lambda\tau}g_{\mu\rho} - g_{\lambda\rho}g_{\mu\tau}) \\ Tr(\gamma^\nu \sigma_{\lambda\mu}\gamma^\eta \sigma_{\tau\rho}) = 4\delta^\nu{}_\rho(g_{\mu\tau}\delta_\lambda{}^\eta - \delta_\mu{}^\eta g_{\lambda\tau}) + 4\delta^\nu{}_\tau(g_{\lambda\rho}\delta_\mu{}^\eta - \delta_\lambda{}^\eta g_{\mu\rho}) \\ + 4g^{\nu\eta}(g_{\lambda\tau}g_{\mu\rho} - g_{\lambda\rho}g_{\mu\tau}) + 4\delta_\mu{}^\nu(g_{\lambda\rho}\delta^\eta{}_\tau - g_{\lambda\tau}\delta^\eta{}_\rho) \\ + 4\delta^\nu{}_\lambda(g_{\mu\tau}\delta^\eta{}_\rho - g_{\mu\rho}\delta^\eta{}_\tau). \tag{2.35}$$

After evaluation of the traces, the cross-product is found to be

$$\frac{1}{64\pi^2 s} \frac{e_0^6}{64 sin^6\theta_W cos^6\theta_W} \frac{16(q^2 - M_Z^2)[g_{Ve}^{(0)2} - g_{Ae}^{(0)2}]^2}{((q^2 - M_Z^2)^2 + \frac{1}{4}\Gamma_Z^2 M_Z^2)^2} \tilde{\Delta}_2(q^2)^2 \left(\frac{2q^2}{m_e^2} + 2\right) \\ \left\{p_1 \cdot p_3 p_2 \cdot p_4 q^2 - p_1 \cdot q p_2 \cdot q p_2 \cdot p_4 + p_2 \cdot p_3 p_1 \cdot p_4 q^2 - p_2 \cdot q p_3 \cdot q p_1 \cdot p_4 \right. \\ + p_1 \cdot p_4 p_2 \cdot p_3 q^2 - p_1 \cdot q p_4 \cdot q p_2 \cdot p_3 + p_2 \cdot p_4 p_1 \cdot p_3 q^2 - p_2 \cdot q p_4 \cdot q p_1 \cdot p_3 \\ - 2(p_1 \cdot p_2 + m_e^2)(p_3 \cdot p_4 q^2 - p_3 \cdot q p_4 \cdot q) - 2(p_3 \cdot p_4 + m_e^2)(p_1 \cdot p_2 q^2 - p_1 \cdot q p_2 \cdot q) \\ \left. + 3(p_1 \cdot p_2 + m_e^2)(p_3 \cdot p_4 + m_e^2) q^2 \right\}. \tag{2.36}$$

The ratio of this term to the first term in the expansion of the differential cross-section is

$$\frac{e_0^2}{4sin^2\theta_W cos^2\theta_W}\frac{q^2-M_Z^2}{(q^2-M_Z^2)^2+\frac{1}{4}\Gamma_Z^2M_Z^2}\cdot\tilde{\Delta}_2(q^2)^2\cdot\frac{q^2+m_e^2}{m_e^2}$$
$$\frac{1}{(p_1\cdot p_3p_2\cdot p_4+p_1\cdot p_4p_2\cdot p_3+2m_e^2)}\Big\{p_1\cdot p_3p_2\cdot p_4q^2-p_1\cdot qp_3\cdot qp_2\cdot p_4$$
$$+p_2\cdot p_3p_1\cdot p_4q^2-p_2\cdot qp_3\cdot qp_1\cdot p_4+p_1\cdot p_4p_2\cdot p_3q^2-p_1\cdot qp_4\cdot qp_2\cdot p_3$$
$$+p_2\cdot p_4p_1\cdot p_3q^2-p_2\cdot qp_4\cdot qp_1\cdot p_3-2(p_1\cdot p_2+m_e^2)(p_3\cdot p_4q^2-p_3\cdot qp_4\cdot q)$$
$$-2(p_3\cdot p_4+m_e^2)(p_1\cdot p_2q^2-p_1\cdot p_2\cdot q)+3(p_1\cdot p_2+m_e^2)(p_3\cdot p_4+m_e^2)q^2\Big\}. \tag{2.37}$$

Since this expression vanishes at $q^2=M_Z^2$, it will be augmented by higher-loop vacuum polarization graphs yielding

$$\sum_{n=2}^{\infty}(-1)^n\frac{(q^2)^{n-2}\tilde{\Pi}^c(q^2)^{N-2}\cdot 2\ cos\ (n-1)\alpha}{[(q^2-M_Z^2)^2+\frac{1}{4}\Gamma_Z^2M_Z^2][(q^2-M_Z^2)^2+\frac{1}{4}\Gamma_Z^2M_Z^2]^{\frac{n-1}{2}}} \tag{2.38}$$

in place of

$$\frac{1}{q^2-M_Z^2+\frac{i}{2}M_Z\Gamma_Z}\frac{1}{\left(q^2-M_Z^2-\frac{i}{2}\Gamma_ZM_Z\right)^2}+\frac{1}{\left(q^2-M_Z^2+\frac{i}{2}\Gamma_ZM_Z\right)^2}\frac{1}{q^2-M_Z^2-\frac{i}{2}\Gamma_ZM_Z}. \tag{2.39}$$

The new expression equals

$$2\frac{(1+\tilde{\Pi}^c(q^2))q^2-M_Z^2}{\left[(q^2-M_Z^2)^2+(\Pi^c(q^2)^2+2\tilde{\Pi}^c(q^2))q^4+M_Z^4\left(\frac{1}{4}\Gamma_Z^2-q^2\tilde{\Pi}^c(q^2)\right)\right]}. \tag{2.40}$$

The average of the momentum expression would be

$$\frac{4\left(1-\frac{1}{3}\right)+2\left(-2+\frac{2}{3}\right)+3}{2}q^2=\frac{3}{2}q^2. \tag{2.41}$$

Since

$$\frac{3\tilde{\Pi}^c(M_Z^2)}{3\tilde{\Pi}^c(M_Z^2)\left(1+3\tilde{\Pi}^c(M_Z^2)\right)+\frac{1}{4}\frac{\Gamma_Z^2}{M_Z^2}}\approx 0.9729\pm 0.0172 \tag{2.42}$$

the ratio has the following value

$$(0.1287)\cdot 3\cdot(0.9729\pm 0.0172)\cdot\frac{M_Z^2+m_e^2}{m_e^2}\cdot\tilde{\Delta}_2(M_Z^2)^2. \tag{2.43}$$

It is known that $\tilde{\Delta}_2(0) = \frac{e^2}{8\pi^2}$, and that it scales with q^2 approximately as $\frac{m_e^2}{q^2}$ for $|\vec{q}|^2 \gg m_e^2$ [7]. The factor of $\tilde{\Delta}_2(q^2)^2$ can be derived from a convolution of integrals. This enfolding would consist of one Z^0 propagator, whereas the expression for the amplitude should contain two Z^0 propagators. The inclusion of an additional propagator increases the result by $\frac{M_Z^2}{m_e^2}$ approximately at $q^2 = M_Z^2$. To close the loop in the convolution, one additional fermion propagator, to join the Z^0 propagators, is required to generate the amplitude. The removal of one fermion propagator again produces a momentum-dependent factor, which equals nearly $\frac{M_Z}{m_e}$ at $q^2 = M_Z^2$. The coefficient of the identity matrix in $[g_{Ve}^{(0)} - \gamma_5 g_{Ae}^{(0)}]^2$, $g_{Ve}^{(0)2} + g_{Ae}^{(0)2}$, equals 0.2526626 ± 0.000304, yielding

$$\begin{aligned}(0.1287) \cdot 3 \cdot (0.9729 \pm 0.0172)\frac{M_Z}{m_e} \cdot \left((0.2526626 \pm 0.000304)\frac{e^2}{8\pi^2}\right)^2 \\ \approx (5.77517 \pm 0.22484) \times 10^{-3}.\end{aligned} \tag{2.44}$$

The ratios for the processes with $\mu^+\mu^-$ and $\tau^+\tau-$ in the final state would be considerably reduced, such that the sum would be $(5.80476 \pm 0.22599) \times 10^{-3}$. There are other known two-loop diagrams, which either cancel by the earlier computation for the self-energy subgraphs or may be included in the auxiliary sum over vacuum polarization subgraphs [8].

From this percentage of the cross-section, the contribution to the width of the Z^0 resonance may be estimated.

$$\begin{aligned}\Delta\Gamma_{two-loop} &= ((5.80476 \pm 0.22599) \times 10^{-3})((2.4952 \pm 0.0023)\ GeV) \\ &\doteq (14.48404 \pm 0.56392)\ MeV.\end{aligned} \tag{2.45}$$

Addition of the widths (2.28) and (2.45) gives $(490.1714 \pm 13.2011)\ MeV$, which nearly equals the invisible width of the Z^0 resonance, $499.0 \pm 1.5\ MeV$ [3][9][10], that has been attributed previously to neutrinos.

3. The Existence of New Particles in the Final State

The sum of the partial widths, including the decays $Z^0 \rightarrow \gamma\gamma$ and $Z^0 \rightarrow e^+e^-e^+e^-$, equals nearly the total width, which has implications for the existence of supersymmetric partners of the known particles. For the W boson, the reaction $W^- \rightarrow e^-\bar{\nu}$ has the supersymmetric analogue $W^- \rightarrow e_s^-\bar{\nu}_s$, and the partial width is given by

$$\Gamma_{W^- \rightarrow e_s^-\bar{\nu}_s} = \frac{1}{2}\left[\left(1 - \frac{M_{\nu_s}^2}{M_W^2} - \frac{M_{e_s}^2}{M_W^2}\right)^2 - 4\frac{M_{\tilde{n}u_s}M_{e_s}^2}{M_W^4}\right]^{\frac{3}{2}}\Gamma_{W^- \rightarrow e^-\bar{\nu}} \tag{3.1}$$

since the formula [11] follows from the charge conjugate of the equation for the decay $W^+ \rightarrow e_s^+ \nu_s$ with the masses being invariant. If $M_{\bar{\nu}_s}$, $M_{e_s} \rightarrow 0$, $\Gamma_{W^- \rightarrow e_s^- \bar{\nu}_s} \rightarrow \frac{1}{2}\Gamma_{W^- \rightarrow e^- \bar{\nu}}$ are excluded.

For the Z^0 boson, the supersymmetric analogue of the decay mode $Z^0 \rightarrow \nu\bar{\nu}$ is $Z^0 \rightarrow \nu_s\bar{\nu}_s$ [12],

$$\Gamma_{Z^0 \rightarrow \nu_s \bar{\nu}_s} = 80 \left(\frac{v_{\nu_s}}{c}\right)^3 MeV. \tag{3.2}$$

It can be verified that the sum of the partial widths is

$$(\Gamma_{Z^0 \rightarrow e^+e^-} + \Gamma_{Z^0 \rightarrow \mu^+\mu^-} + \Gamma_{Z^0 \rightarrow \tau^+\tau^-})_{known} + \Gamma_{invisible} + \Gamma_{Z^0 \rightarrow hadrons} = (2.495352 \pm 0.002504)\ GeV. \tag{3.3}$$

Since this coincides with the total width within the experimental error, the only a fraction of the invisible width could result from any new particles. Since the experimental uncertainty in the difference between the sum of the partial widths and the total width would be 3.4 MeV, it can be combined with the experimental error in $\Gamma_{invisible}$ is 1.5 MeV to give an estimated bound of 3.8678 MeV for the contribution of new particles can be used. Then $\frac{v_{\nu_s}}{c}$ should be less than 0.365. The mass ν_s then must exceeed $\frac{91.1786\ GeV/c^2}{2\gamma_{\nu_s,max}} = 42.46\ GeV/c^2$ when the center-of-mass energy is shared equally between ν_s and $\bar{\nu}_s$.

Similarly, there is the reaction $Z^0 \rightarrow e_s^+ e_s^-$, which would be considered to be an intermediate stage before the final decay to e^+e^- [12].

$$\Gamma_{Z^0 \rightarrow e_s^+ e_s^-} = \frac{1}{2}\left(\frac{2p_{e_s}}{M_Z}\right)^3 \Gamma_{Z^0 \rightarrow e^+e^-}. \tag{3.4}$$

Therefore, the uncertainties related to the difference between the sum of the partial widths and the total width and $\Gamma_{Z^0 \rightarrow e^+e^-}$ are combined. As the higher-loop contributions to this width can be included, the value of

$$83.984\ MeV + \frac{1}{3}(475.6874\ MeV) + \frac{1}{1.00512392}(499\ MeV - 475.6874\ MeV) \doteq 268.284\ MeV \tag{3.5}$$

may be used in Eq.(3.4). Based on one third of the uncertainty in the invisible width and that of the width for the known decays of Z^0 to e^+e^-, the maximum contribution of $e_s^+e_s^-$ pairs to the width would be 3.4376 MeV approximately. The momentum then must satisfy $p_{e_s} < 2^{\frac{1}{3}}(0.0128)^{\frac{1}{3}}\frac{M_Z c}{2} \doteq 13.44222\ (GeV/c)$, yielding a minimum of 43.5672 GeV/c^2 for the mass M_{e_s}.

Searches for new particles of this kind through missing energy and momentum in the production of leptons and hadronic jets have provided a lower bound 71.3 GeV, although the centre-of-mass energies of the e^+e^- collisions over 192 GeV are considerably higher than $M_Z c^2$ [13]. In these experiments, if the Z^0 boson was carrying an energy larger than 180 GeV, the distribution, and therefore, the total and partial widths of the plot centered at this energy, would be increased by a factor $\gamma_{Z^0} \geq 1.97368421$.

For the reaction $Z^0 \rightarrow \nu_s \bar{\nu}_{\nu_s}$,

$$80 \left(\frac{\tilde{v}_{\nu_s}}{c} \right)^3 MeV = 1.97368421 \cdot (3.8678\ MeV) \tag{3.6}$$

and

$$\frac{\tilde{v}_{\nu_s}}{c} = 0.456966 \tag{3.7}$$

such that

$$M_{\nu_s} > \frac{180\ GeV/c^2}{2\tilde{\gamma}_{\nu_s}} \simeq 80.0535\ GeV/c^2. \tag{3.8}$$

The formula for the partial width of the decay $Z^0 \rightarrow e_s^+ e_s^-$ would imply that the upper bound for p_{e_s} is unchanged because the Lorentz factors cancel in the ratio $\frac{\Gamma_{Z^0 \rightarrow e_s^+ e_s^-}}{\Gamma_{Z^0 \rightarrow e^+ e^-}}$. If the center-of-mass energy is shared equally by e_s^+ and e_s^-, then

$$M_{e_s} > 88.99049\ GeV/c^2 \tag{3.9}$$

which is larger than the previous bound. It would not be likely that an inequality of this type could be derived for arbitrarily large Z^0 energies, because the validity of Eq.(3.4) may be restricted by unitarity bounds. The experimental search for new particles belonging to massive supermultiplets and the coincidence of the sum of the partial widths of the Z^0 decay into known particles and the total width provide support for the breaking of supersymmetry at the energy levels of the masses of the standard particle spectrum.

Supersymmetry is likely not to be present at energies significantly lower than the unification scale, because the relation between the gauge coupling and string tension can be shown to be no longer valid [14]. The absence of supersymmetry in a low-energy ground state is an indication of either the non-existence of supersymmetric partners of the known particles at lower energies or the occurrence of new particles at energies closer to the grand unified scale. The viability of a supersymmetric solution at higher energy scales is consistent, therefore, with the theoretical study of the Z^0 resonance.

4. Conclusion

With a propagator for the Z^0 boson defined such that the decay of this particle is reflected in the imaginary part of the pole in the momentum plane, a formula for the differential cross section consisting of the leading term and higher-loop vacuum polarization diagrams, is given. After the computation of several two-loop diagrams, the sum of the higher-loop widths can be compared to the invisible width of the Z^0 resonance, and it is established that these values coincide within the theoretical uncertainty derived from $\tilde{\Pi}^c$, M_Z and Γ_Z. Since it is expected that the remaining width shall result mainly from other types of diagrams with three or more loops, the experimental limits rather than the theoretical uncertainty are used in determining an upper bound for the difference between the sum of the partial widths for decays into known particles and the total width. From this value, lower bounds for the masses of new particles belonging to supermultiplets are derived.

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HECKMANN'S WORMHOLES IN JORDAN-BRANS-DICKE GRAVITY

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Abstract

A simple Heckmann's vacuum wormhole solution of Jordan-Brans-Dicke gravitation is presented and analysed. It is shown that in contrast with class I Brans solution where the throat radius becomes real when $\omega < -4/3$ here it becomes positive when $\omega < -1$.

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1 Introduction

Lately there has been renewed interest in the scalar-tensor theories of gravitation. Important arena where these theories have found immense applications is the field of wormhole physics. Several classes of solutions in the scalar-tensor theories support wormhole geometry. The most prominent example of scalar-tensor theories is perhaps the Jordan-Brans-Dicke (JBD) [1], [2]. Now a day, predictions of JBD theory to be consistent not only with the weak field solar system tests [3], [4] but also with the recent cosmological observations [5].

JBD theory describes gravitation through a metric tensor $g_{\mu\nu}$ and a massless scalar field ϕ. In this theory, static wormhole solutions were found in vacuum, the source of gravity being the scalar field. Several static wormhole solutions in JBD theory have been widely investigated in the literature [6],[7], [8]. It was shown that three of the four Brans classes of vacuum solutions admit a wormholelike spacetime for convenient choices of their parameters.

In what follows, we shall present a static vacuum wormhole solution of the BD theory endowed with first exact solution of JBD field equations were obtained in parametric form by Heckmann [9], soon after Jordan proposed scalar tensor theory.

2 The Heckmann's wormholes solutions.

JBD theory are described by the following action in the Jordan frame is:

$$S = \int dx\sqrt{-g}(\phi R - \omega g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \\ ) + S_m. \quad (1)$$

Here, R is the Ricci scalar curvature with respect to the space-time metric $g_{\mu\nu}$ and S_m denote action of matter fields. (We use units in which gravitational constant G=1 and speed of light c=1.)

Variation of (1) with respect to $g_{\mu\nu}$ and ϕ gives, respectively, the field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{\phi}T^M_{\mu\nu} + T^{JBD}_{\mu\nu}, \quad (2)$$

where

$$T_{\mu\nu}^{JBD} = [\frac{\omega}{\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\phi\nabla^\alpha\phi\right) + \\ + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla_\alpha\nabla^\alpha\phi\right)]. \tag{3}$$

and

$$\nabla_\alpha\nabla^\alpha\phi = \frac{T_\lambda^{M\ \lambda}}{3+2\omega}, \tag{4}$$

and $T_\lambda^{M\ \lambda}$ is the energy momentum tensor of ordinary matter which obeys the conservation equation $T_{\mu\nu;\lambda}^{M}\ g^{\nu\lambda}= 0$.

One can chose the static spherically symmetric metric in curvature coordinates form

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2d\Omega^2. \tag{5}$$

Then the solutions of the gravitational field equations in the vacuum take the form [9]

$$r = \frac{\alpha_0}{\sqrt{\tau}\left(\tau^{-h} - \tau^h\right)}, \\ e^\lambda = \frac{4h^2}{\left[\left(\frac{1}{2}+h\right)\tau^h - \left(\frac{1}{2}-h\right)\tau^{-h}\right]^2}, \\ e^\nu = \tau^{\frac{1}{B}}, \tag{6} \\ \phi = \phi_0\tau^{\frac{\beta_0}{B}},$$

where τ parameter, arbitrary constant and

$$h^2 = \frac{1}{4} - \frac{A}{B^2};\ A = \frac{\beta_0}{2}\left(1-\beta_0\omega\right);\ B = 1+2\beta_0.$$

The constant ϕ_0 are determined by an asymptotic flatness condition as ϕ_0 =1, while α_0 and β_0 is determined by the requirement of having Schwarzschild geometry in the weak field limit [1]. Thereby

$$\alpha_0 = 4hM\left(1+2\beta_0\right)$$

is the function of central mass of the configuration and

$$\beta_0 = -\frac{1}{3+2\omega}. \tag{7}$$

This implies that the range of β_0 is dictated by the range of ω, which, in turn, is to be dictated by the requirements of wormhole geometry. In order

to investigate whether a given solution represents a wormhole geometry, it is convenient to cast the metric into Morris-Thorne canonical form:

$$ds^2 = -e^{2\chi\left(\overset{*}{R}\right)}dt^2 + \left[1 - \frac{b\left(\overset{*}{R}\right)}{\overset{*}{R}}\right]^{-1} dr^2 + \overset{*}{R}^2 d\Omega^2, \tag{8}$$

where $\chi\left(\overset{*}{R}\right)$ and $b\left(\overset{*}{R}\right)$ are arbitrary functions of the radial coordinate, $\overset{*}{R}$. $\chi\left(\overset{*}{R}\right)$ is denoted as the redshift function, for it is related to the gravitational redshift; $b\left(\overset{*}{R}\right)$ is called the form function, because as can be shown by embedding diagrams, it determines the shape of the wormhole [10]. The radial coordinate has a range that increases from a minimum value at $\overset{*}{R}_0$, corresponding to the wormhole throat, to infinity. The Heckmann solution can be cast to the form (8) by defining a radial coordinate $\overset{*}{R}$ which is related with r via the expression

$$\overset{*}{R} = \frac{\alpha_0}{\sqrt{\tau}\left(\tau^{-h} - \tau^{h}\right)}, \tag{9}$$

The functions $\chi\left(\overset{*}{R}\right)$ and $b\left(\overset{*}{R}\right)$ are the given by

$$\chi\left(\overset{*}{R}\right) = \tau^{\frac{1}{B}}, \tag{10}$$

$$b\left(\overset{*}{R}\right) = \overset{*}{R}\left[(1 - [\frac{(1+2h)}{4h}\tau^{h} - \frac{(1-2h)}{4h}\tau^{-h}]^2)\right]$$

The axially symmetric embedded surface $z = z\left(\overset{*}{R}\right)$ shaping the wormhole's spatial geometry is obtained from

$$\frac{dz}{d\overset{*}{R}} = \pm\left[\frac{\overset{*}{R}}{b\left(\overset{*}{R}\right)} - 1\right]^{-\frac{1}{2}} \tag{11}$$

By definition of wormhole at throat its embedded surface is vertical. The throat of the wormhole occurs at $\overset{*}{R} = \overset{*}{R}_0$ such that $b\left(\overset{*}{R}_0\right) = \overset{*}{R}_0$. This gives minimum allowed R - coordinate radii $\mathrm{R}_0^{\pm}$ as [9]

$$R_0^{\pm} = \frac{\alpha_0\sqrt{1-4h^2}}{4h}\left(\frac{1+2h}{1-2h}\right)^{\frac{1}{4h}} \tag{12}$$

The values $\mathrm{R}_0^{\pm}$ can be obtained from (13) using the (7).

$$R_0^{\pm} = \sqrt{\frac{-3\left(1+\omega\right)}{2}}\left(\frac{1+2\omega+\Omega}{1+2\omega-\Omega}\right)^{\frac{1+2\omega}{\Omega}}, \tag{13}$$

where $\Omega = \sqrt{7 + 10\omega + 4\omega^2}$.

In contrast with class I Brans solution where the throat radius becomes real when $\omega < -4/3$ here it becomes positive when $\omega < -1$. This range thus gives rise viable wormhole geometry. The redshift function has a singularity at $\tau = 0$ which corresponds to the point $\overset{*}{R}= 0$.

This solution is regular in the range $0 < \tau < 1$, does not contain horizons and is asymptotically flat as $\tau \to 1$. However, in the case $\tau \to 0$ emerges as the extremely non Euclidian asymptotic. The important point is that the Heckmann solutions depend on the parameter h. This implies that the range of ω is dictated by the range of h, which, in turn, is to be dictated by the requirements of wormhole geometry. Redefining the parameter h as $\widehat{h}^2 = -h^2$ we obtain the following solution [9]

$$r = \frac{r_0}{2\sqrt{\tau}\sin\left(\widehat{h}\ln\tau\right)}, \tag{14}$$

$$e^{\lambda} = \frac{4\widehat{h}^2}{\left[2\widehat{h}\cos\left(\widehat{h}\ln\tau\right) + \sin\left(\widehat{h}\ln\tau\right)\right]^2},$$

$$e^{\nu} = \tau^{\frac{1}{B}}, \tag{15}$$

$$\phi = \phi_0\tau^{\frac{\beta_0}{B}},$$

then function $b\left(\overset{*}{R}\right)$ is the given by

$$b\left(\overset{*}{R}\right) = \overset{*}{R}\left[(1 - \frac{\left[2\widehat{h}\cos\left(\widehat{h}\ln\tau\right) + \sin\left(\widehat{h}\ln\tau\right)\right]^2}{4\widehat{h}^2})\right]. \tag{16}$$

This gives minimum allowed R - coordinate radii $\mathrm{R}_0^{\pm}$ as

$$R_0^{\pm} = \frac{r_0\sqrt{\widehat{h}^3}\exp\frac{\arccos\left(-\frac{1}{\sqrt{1+4\widehat{h}^2}}\right)}{2\widehat{h}}}{\arccos\left(-\frac{1}{\sqrt{1+4\widehat{h}^2}}\right) - \widehat{h}\ln\widehat{h}},$$

or

$$R_0^{\pm} = \frac{r_0\sqrt{\widehat{h}^3}\exp\frac{\arctan(2\widehat{h})}{2\widehat{h}}}{\arctan\left(2\widehat{h}\right) - \widehat{h}\ln\widehat{h}}.,$$

3 Discussion

The field equations in JBD gravity theory are non linear in nature. Moreover, the physical and the geometrical meaning of the radial coordinate r

are not defined by symmetry reasons and are unknown a priori. In this context, using the key assumption that the Heckmann [9] solution physically acceptable it was constructed the spherically symmetric wormholes.

Our analysis reveals that ω may take on arbitrary negative values less then -1. This result extends the scope for the feasibility of wormhole scenarios even to the regime of ordinary observations.

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QUANTUM ALGEBRA OF THE HAMILTONIAN CONSTRAINT IN REDUCED 4-DIMENSIONAL GRAVITY

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Abstract

In this paper we demonstrate closure of the quantum algebra of Hamiltonian constraints in a theory directly related to a certain sector of general relativity reduced to diagonal variables.

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1 Introduction

In [1] an action was presented which is related to a certain sector of reduced gravity, and it was demonstrated that the associated system is Dirac consistent at the classical level as a stand-alone action. By reduced gravity, we mean that the theory has three degrees of freedom per point at the kinematic level, namely the level prior to implementation of the Hamiltonian constraint. In this theory there are no Gauss' law and no diffeomorphism constraints. Additionally, there are six distinct sectors, referred to as quantizable configurations Γ_q, which exhibit the same features as outlined in [1]. The results of the present paper will also apply to these configurations Γ_q. The purpose of the present paper will be to verify closure of the algebra of constraints for the action written down in [1] at the quantum level. In this section we will introduce the action, and in section 2 we carry out the computation of the quantum constraints algebra.

Let us consider a system with configuration and momentum space variables $\Gamma_{Kin} = (X, Y, T)$ and $P_{Kin} = (\Pi_1, \Pi_2, \Pi)$ defined on a 4-dimensional spacetime manifold of topology $M = \Sigma \times R$, where Σ is a 3-dimensional spatial hypersurface. The variables are in general complex, and the configuration space variables take on the ranges $-\infty < |X|, |Y|, |T| < \infty$. The mass dimensions of all variables have been chosen to be

$$[\Pi_1] = [\Pi_2] = [\Pi] = 1; \quad [X] = [Y] = [T] = 0. \tag{1}$$

These variables define the following kinematic phase space action for a totally constrained system

$$I_{Kin} = -\frac{i}{G}\int dt \int_\Sigma d^3x\big(\Pi_1\dot{X} + \Pi_2\dot{Y} + \Pi\dot{T}\big) - iH[N]. \tag{2}$$

The field N is an auxilliary field smearing a phase space function H, such that the Hamiltonian density is given by

$$H[N] = \int_\Sigma d^3x N U e^{-T/2}\Phi. \tag{3}$$

The quantities in (3) are defined as follows. First we have Φ, given by

$$\Phi = \sqrt{\Pi(\Pi+\Pi_1)(\Pi+\Pi_2)}\Big[\Big(k + e^T\Big(\frac{1}{\Pi} + \frac{1}{\Pi+\Pi_1} + \frac{1}{\Pi+\Pi_2}\Big)\Big] \tag{4}$$

where k is a numerical constant.[1] There are no spatial derivatives in any of the quantities in (4), and all spatial derivatives in the theory (2) are confined

[1] For the reduced sector of gravity, we must have $k = \Lambda a_0^{-3}$, where Λ is the cosmological constant and a_0 is a numerical constant of mass dimension $[a_0] = 1$.

to the quantity U, given by

$$U = \Big[1 + e^{-T}\big((\partial_2 Z)(\partial_3 X)(\partial_1 Y) - (\partial_3 Y)(\partial_1 Z)(\partial_2 X)\big)$$
$$+e^{-2X}(\partial_1 Y)(\partial_1 Z) + e^{-2Y}(\partial_2 Z)(\partial_2 X) + e^{-2Z}(\partial_3 X)(\partial_3 Y)\Big]^{1/2} \tag{5}$$

with $Z = T - X - Y$. We have defined

$$\partial_1 = \frac{\partial}{\partial y^1};\quad \partial_2 = \frac{\partial}{\partial y^2};\quad \partial_3 = \frac{\partial}{\partial y^3}, \tag{6}$$

where y^1, y^1 and y^3 are dimensionless spatial coordinates in Σ.

The canonical structure of (2) yields the following fundamental Poisson brackets

$$\{X(x,t), \Pi_1(y,t)\} = \{Y(x,t), \Pi_2(y,t)\} = \{T(x,t), \Pi(y,t)\} = -iG\delta^{(3)}(x,y). \tag{7}$$

In this paper we will check for closure of the quantum constraints algebra of (2). But prior to proceeding with the algebra, it is worthwhile to present a short background of the significance of the action (2).

1.1 Relation to an antecedent of the CDJ action

The significance of the action (2) is that it can be obtained from a restricted sector of an action for general relativity, which appears in [2] as an intermediate step in obtaining the CDJ pure spin connection formulation for gravity,[2] as we will demonstrate. Consider the following change of variables of the action (2)

$$\Pi = a_0^3 e^T \lambda_3;\quad \Pi + \Pi_1 = a_0^3 e^T \lambda_1;\quad \Pi + \Pi_2 = a_0^3 e^T \lambda_2 \tag{8}$$

for the momentum space variables P_{Kin}, and

$$X = \ln\Big(\frac{a_1}{a_0}\Big);\quad Y = \ln\Big(\frac{a_2}{a_0}\Big);\quad T = \ln\Big(\frac{a_1 a_2 a_3}{a_0^2}\Big) \tag{9}$$

for the configuration space variables Γ_{Kin}, where a_0 is a numerical constant of mass dimension $[a_0] = 1$. Let us also make the definitions

$$x^1 = \frac{y^1}{a_0};\quad x^2 = \frac{y^2}{a_0};\quad x^3 = \frac{y^3}{a_0} \tag{10}$$

with y^1, y^2 and y^3 the dimensionless spatial coordinates in Σ. This implies that $[x^1] = [x^2] = [x^3] = -1$, namely that the coordinates x^1, x^2 and x^3

[2]The initial CDJ refer to Capovilla, Dell and Jacobson, who developed a nonmetric formulation for gravity written almost completely in terms of the spin connection.

have dimensions of length. Substitution of (9) and (10) into (5) yields

$$U = (a_1a_2a_3)^{-1}\Big[(a_1a_2a_3)^2 + (\partial_2a_3)(\partial_3a_1)(\partial_1a_2)$$
$$-(\partial_3a_2)(\partial_1a_3)(\partial_2a_1) + a_2a_3(\partial_1a_2)(\partial_1a_3) + a_3a_1(\partial_2a_3)(\partial_2a_1)$$
$$+a_1a_2(\partial_3a_1)(\partial_3a_2)\Big]^{1/2} = (\det A)^{-1}(\det B)^{1/2}, \qquad (11)$$

from which one recognizes U as the square root of the determinant of the magnetic field B^i_a for a diagonal connection $A^a_i = diag(a_1, a_2, a_3)$, with the leading order term in $(\det A)$ factored out. In matrix form this is given by

$$a^a_i = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}; \quad b^i_a = \begin{pmatrix} a_2a_3 & -\partial_3a_2 & \partial_2a_3 \\ \partial_3a_1 & a_3a_1 & -\partial_1a_3 \\ -\partial_2a_1 & \partial_1a_2 & a_1a_2 \end{pmatrix}.$$

Substitution of (8), (9) and (11) into (2) yields

$$I = -\frac{i}{G}\int dt \int_\Sigma d^3x\Big(\lambda_1a_2a_3\dot{a}_1 + \lambda_2a_3a_1\dot{a}_2 + \lambda_3a_1a_2\dot{a}_3$$
$$-iN(\det b)^{1/2}\sqrt{\lambda_1\lambda_2\lambda_3}\Big(\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\Big). \qquad (12)$$

Let us make the following definitions for the magnetic field and the temporal component of the curvature

$$b^i_a = \frac{1}{2}\epsilon^{ijk}f^a_{jk}; \quad f^a_{0i} = \dot{a}^a_i - D_ia^a_0, \qquad (13)$$

where D_i is the $SO(3,C)$ covariant derivative with respect to the spatial connection a^a_i. Then the integrand of the canonical one form of (12) can be written as

$$\lambda_gb^i_g\dot{a}^g_i = \frac{1}{2}\lambda_g\epsilon^{ijk}f^g_{jk}f^g_{0i} + \lambda_gb^i_gD_ia^g_0, \qquad (14)$$

where a^g_0 is the temporal component of the connection a^g_μ. Then defining $\epsilon^{ijk} = \epsilon^{0ijk}$ and using the symmetries of the 4-D epsilon symbol $\epsilon^{\mu\nu\rho\sigma}$, then (14) is given by

$$\frac{1}{8}\lambda_gf^g_{\mu\nu}f^g_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} - a^g_0b^i_gD_i\lambda_g. \qquad (15)$$

Using equation (15) to replace the canonical one form in (12), we get the action

$$I = -\frac{i}{G}\int_M d^4x\Big[\frac{1}{8}\lambda_g f^g_{\mu\nu}f^g_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} -i\eta\Big(\Lambda+\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}\Big)\Big] + \int dt\int_\Sigma d^3x a^g_0 b^i_g D_i\lambda_g, \tag{16}$$

where $\eta = (\det b)^{1/2}\sqrt{\lambda_2\lambda_2\lambda_3}$. Equation (16) is none other than the CDJ antecedent appearing in [2] with the following caveats: (i) The Gauss' law constraint is missing. This is the last term on the right hand side of (16), which cancels the same quantity from the curvature squared term. (ii) Equation (16) is the restriction of the aforementioned action to diagonal variables.

2 Quantum constraints algebra of the Hamiltonian constraint

Upon quantization of (2), the dynamical variables become promoted to quantum operators satisfying equal-time commutation relations

$$\big[\hat{X}(x,t),\hat{\Pi}_1(y,t)\big] = \big[\hat{Y}(x,t),\hat{\Pi}_2(y,t)\big] = \big[\hat{T}(x,t),\hat{\Pi}(y,t)\big] = (\hbar G)\delta^{(3)}(x,y), \tag{17}$$

with all other commutators vanishing. The smeared Hamiltonian constraint (3) becomes promoted to a composite operator constraint

$$\hat{H}[N] = \int_\Sigma d^3x N(x)\hat{\eta}(x)\hat{\Phi}(x), \tag{18}$$

where we have made the following definitions

$$\hat{\eta}(x) = \hat{U}(x)e^{-\hat{T}(x)/2};$$
$$\hat{\Phi} = \sqrt{\hat{\Pi}(\hat{\Pi}+\hat{\Pi}_1)(\hat{\Pi}+\hat{\Pi}_2)}\Big[k+\Big(\frac{1}{\hat{\Pi}}+\frac{1}{\hat{\Pi}+\hat{\Pi}_1}+\frac{1}{\hat{\Pi}+\hat{\Pi}_2}\Big)e^{\hat{T}}\Big], \tag{19}$$

with the operator ordering as indicated. The physical states are defined as those states $|\psi\rangle \in |\psi_{Phys}\rangle$ such that $\hat{H}|\psi\rangle = 0$ with $\hat{\Phi}$ appearing to the right. For the quantum constraints algebra to be consistent in the Dirac sense, the algebra must close with $\hat{\Phi}$ appearing on the right.

References [4] and [5]) state that the quantization of theories containing operator products evaluated at the same point results in infinities which need to be regularized. A possible regularization prescription is to individually smear each operator appearing in the operator product. We will show that such regularization procedures are not necessary for the case presented in this paper, since the smearing of the constraints automatically eliminates any infinitites. For the quantum constraints algebra we will use the following operator identity for composite operators

$$[\hat{A}\hat{B},\hat{C}\hat{D}] = \hat{C}[\hat{A},\hat{D}]\hat{B} + \hat{A}[\hat{B},\hat{C}]\hat{D} + [\hat{A},\hat{C}]\hat{B}\hat{D} + \hat{C}\hat{A}[\hat{B},\hat{D}], \qquad (20)$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are bosonic operators. Using equation (20), the quantum constraints algebra of the Hamiltonian (18) is given by

$$[\hat{H}[M],\hat{H}[N]] = \int_\Sigma d^3x \int_\Sigma d^3y M(x)N(y)[\hat{\eta}(x)\hat{\Phi}(x),\hat{\eta}(y)\hat{\Phi}(y)]$$

$$= \int_\Sigma d^3x \int_\Sigma d^3y M(x)N(y)\Big[\hat{\eta}(y)[\hat{\eta}(x),\hat{\phi}(y)]\hat{\phi}(x) + \hat{\eta}(y)[\hat{\Phi}(x),\hat{\eta}(y)]\hat{\Phi}(y)$$

$$+[\hat{\eta}(x),\hat{\eta}(y)]\hat{\Phi}(x)\hat{\Phi}(y) + \hat{\eta}(x)\hat{\eta}(y)[\hat{\Phi}(x),\hat{\Phi}(y)]\Big]. \qquad (21)$$

We must now analyse each term appearing in (21). The third term on the right hand side of (21) vanishes since it is a commutator purely between configuration space variables. The fourth term of (21) vanishes, which can be seen as follows. Make the following definitions

$$\hat{W}(x,y) = \hat{\eta}(x)\hat{\eta}(y); \quad \hat{\Phi}(x) \equiv \hat{\chi}(x) + \hat{S}(x)e^{\hat{T}(x)} \qquad (22)$$

where χ and S depend only on momentum space variables, whose specific form can be read off from (19). Note that the following relations hold

$$[\hat{\chi}(x),e^{\hat{T}(y)}] = -[e^{\hat{T}(y)},\hat{\chi}(x)] = \Big(\frac{\partial\chi}{\partial\Pi}\Big)_x e^{\hat{T}(y)}\delta^{(3)}(x,y). \qquad (23)$$

This is a consequence of (17), where Π is the only variable with nonvanishing relations with T. We are now ready to proceed with the fourth term of (21), which is given by

$$\int_\Sigma d^3x \int_\Sigma d^3y \hat{W}(x,y)\Big[\hat{\chi}(x) + \hat{S}(x)e^{\hat{T}(x)},\hat{\chi}(y) + \hat{S}(y)e^{\hat{T}(y)}\Big] \qquad (24)$$

where we have used the definitions (22). Expansion of (24) leads to the following four terms

$$\int_\Sigma d^3x \int_\Sigma d^3y \hat{W}(x,y)\big[\hat{\chi}(x),\hat{\chi}(y)\big] + \int_\Sigma d^3x \int_\Sigma d^3y \hat{W}(x,y)\hat{S}(x)\big[\hat{\chi}(x), e^{\hat{T}(y)}\big]$$
$$+ \int_\Sigma d^3x \int_\Sigma d^3y \hat{W}(x,y)\hat{S}(x)\big[e^{\hat{T}(x)},\hat{\chi}(y)\big] + \int_\Sigma d^3x \int_\Sigma d^3y \hat{W}(x,y)\big[\hat{S}(x)e^{\hat{T}(x)}, \hat{S}(y)e^{\hat{T}(y)}\big], \quad (25)$$

which we will in turn analyse. The first term of (25) vanishes due to vanishing commutation relations between momentum space variables. Using the results of (23), the middle two terms of (25) combine into

$$\int_\Sigma d^3x \int_\Sigma d^3y \hat{W}(x,y)\Big[-\Big(\frac{\partial\hat{\chi}}{\partial\hat{\Pi}}\Big)_x e^{\hat{T}(y)} + \Big(\frac{\partial\hat{\chi}}{\partial\hat{\Pi}}\Big)_y e^{\hat{T}(x)}\Big]\delta^{(3)}(x,y)$$
$$= \int_\Sigma d^3x \hat{W}(x,x)\Big[-\Big(\frac{\partial\hat{\chi}}{\partial\hat{\Pi}}\Big)_x e^{\hat{T}(x)} + \Big(\frac{\partial\hat{\chi}}{\partial\hat{\Pi}}\Big)_x e^{\hat{T}(x)}\Big] = 0 \qquad (26)$$

which vanishes after integration of the delta function, leaving remaining the fourth term of (25). Application of the identity (20) to this term yields

$$\int_\Sigma d^3x \int_\Sigma d^3y \hat{W}(x,y)\big[\hat{S}(x)e^{\hat{T}(x)}, \hat{S}(y)e^{\hat{T}(y)}$$
$$= \int_\Sigma d^3x \int_\Sigma d^3y \hat{W}(x,y)\hat{S}(x)\big[\hat{S}(x), e^{\hat{T}(y)}\big]e^{\hat{T}(x)} + \int_\Sigma d^3x \int_\Sigma d^3y \hat{W}(x,y)\hat{S}(x)\big[e^{\hat{T}(x)}, \hat{S}(y)\big]e^{\hat{T}(y}$$
$$+ \int_\Sigma d^3x \int_\Sigma d^3y \hat{W}(x,y)\big[\hat{S}(x),\hat{S}(y)\big]e^{\hat{T}(x)}e^{\hat{T}(y)} + \int_\Sigma d^3x \int_\Sigma d^3y \hat{W}(x,y)\hat{S}(x)\hat{S}(y)\big[e^{\hat{T}(x)}, e^{\hat{T}(y)}\big]$$

The third term of (27) vanishes due to vanishing commutation relations between momentum space variables, and the fourth term vanishes due to vanishing commutation relations between configuration space variables T. Using (23), the first and second term of (27) combine into

$$\int_\Sigma d^3x \int_\Sigma d^3y \hat{W}(x,y)\Big[-\hat{S}(y)\Big(\frac{\partial\hat{S}}{\partial\hat{\Pi}}\Big)_x e^{\hat{T}(x)} + \hat{S}(x)\Big(\frac{\partial\hat{S}}{\partial\hat{\Pi}}\Big)_y e^{\hat{T}(y)}\Big]\delta^{(3)}(x,y)$$
$$= \int_\Sigma d^3x \int_\Sigma d^3y \hat{W}(x,y)\Big[-S\Big(\frac{\partial S}{\partial\Pi}\Big)e^T + S\Big(\frac{\partial S}{\partial\Pi}\Big)e^T\Big] = 0 \quad (28)$$

which also vanishes. We have shown that the third and fourth terms on the right hand side of (21) both vanish, which leaves us with the first and second terms. Make the definitions

$$\frac{\delta\Phi(x)}{\delta\Pi_1(y)} = Q^1(x)\delta^{(3)}(x,y); \quad \frac{\delta\Phi(x)}{\delta\Pi_2(y)} = Q^2(x)\delta^{(3)}(x,y); \quad \frac{\delta\Phi(x)}{\delta\Pi(y)} = Q^3(x)\delta^{(3)}(x,y), \quad (29)$$

where $Q^i(x)$ are functions on Ω_{Kin} whose specific form will not be needed for what follows. Likewise make the definitions

$$\frac{\delta\eta(x)}{\delta X(y)} = \zeta_1^j(x)\delta^{(3)}(x,y)\Big(\frac{\partial}{\partial x^j}\Big);$$
$$\frac{\delta\eta(x)}{\delta Y(y)} = \zeta_2^j(x)\delta^{(3)}(x,y)\Big(\frac{\partial}{\partial x^j}\Big);$$
$$\frac{\delta\eta(x)}{\delta T(y)} = \zeta_3^j(x)\delta^{(3)}(x,y)\Big(\frac{\partial}{\partial x^j}\Big). \tag{30}$$

The notation in (30) signifies that the partial derivatives will act on all objects with x dependence which multiply the terms that the derivatives originally came from. Using (29) and (30), we have the following operator relations

$$\big[\hat{\zeta}(x),\hat{\Phi}(y)\big] = \hat{\zeta}_I^j(x)\hat{Q}^I(y)\delta^{(3)}(x,y)\Big(\frac{\partial}{\partial x^j}\Big);$$
$$\big[\hat{\Phi}(x),\hat{\zeta}(y)\big] = -\hat{\zeta}_I^j(y)\hat{Q}^I(x)\delta^{(3)}(x,y)\Big(\frac{\partial}{\partial y^j}\Big). \tag{31}$$

Hence it is apparent from (31) that $\frac{\partial}{\partial x^i}$ acts on objects containing x dependence, and $\frac{\partial}{\partial y^i}$ acts on objects containing y dependence. So continuing from (21) and using (31), we have

$$\big[\hat{H}[M],\hat{H}[N]\big] =$$
$$\int_\Sigma d^3x\int_\Sigma d^3y M(x)N(y)\Big[\hat{\eta}(y)\hat{\zeta}_I^j(x)\hat{Q}^J(y)\Big(\frac{\partial}{\partial x^j}\Big)\hat{\Phi}(x)$$
$$-\hat{\eta}(x)\hat{\zeta}_I^j(y)\hat{Q}^J(x)\Big(\frac{\partial}{\partial x^j}\Big)\hat{\Phi}(y)\Big]\delta^{(3)}(x,y)$$
$$= \int_\Sigma d^3x\int_\Sigma d^3y\Big[N(y)\hat{\eta}(y)\frac{\partial}{\partial x^j}(M(x)\hat{\zeta}_I^j(x)\hat{Q}^I(y)\hat{\Phi}(x))$$
$$-M(x)\hat{\zeta}(x)\frac{\partial}{\partial y^j}(N(y)\hat{\zeta}_I^j(y)\hat{Q}^I(x)\hat{\Phi}(y))\Big]\delta^{(3)}(x,y). \tag{32}$$

Integration with respect to y collapses the delta function, which yields

$$\int_\Sigma d^3x\Big[N\hat{\zeta}\frac{\partial}{\partial x^j}(M\hat{\zeta}_I^j\hat{Q}^I\hat{\Phi}) - M\hat{\zeta}\frac{\partial}{\partial x^j}(N\hat{\zeta}_I^j\hat{Q}^I\hat{\Phi})\Big]$$
$$= \int_\Sigma d^3x\big(N\partial_i M - M\partial_i N\big)\hat{\eta}\hat{\zeta}_I^j\hat{Q}^I\hat{\Phi}, \tag{33}$$

whence the operator $\hat{\Phi}$ appears to the right. Since $\hat{\Phi}$ is proportional to the Hamiltonian constraint it follows that

$$\big[\hat{H}[M],\hat{H}[N]\big]|\psi\rangle = \hat{H}[M,N]|\psi\rangle, \tag{34}$$

namely that the commutator of two Hamiltonian constraints is a Hamiltonian constraint with the constraint appearing to the right. The quantum algebra of the Hamitonian constraint closes with structure functions, and it closes in direct analogy to its classical counterpart in [1] when one makes the identification $q^I \eta_I^j \rightarrow \hat{\eta} \hat{\zeta}_I^j \hat{Q}^I$. Moreover, the algebra closes with the proper ordering taken into account with the Hamiltonian constraint operator to the right. For these reasons we conclude that the quantum constraints algebra is Dirac consistent and is free of anomalies.

3 Conclusion

The main result of this paper has been to verify the closure of the quantum constraints algebra for a theory of 'reduced' gravity introduced in [1]. Future directions of research will be to investigate the Hibert space structure of the resulting theory.

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ON THE INTEGRAL INVARIANTS OF THE CLOSED RULED SURFACES GENERATED BY A PARALLEL P-ÄQUIDISTANTE DUAL CENTRODE CURVES IN THE LINE SPACE

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Abstract

In this paper, the integral invariants of the closed ruled surfaces generated by parallel p-äquidistante dual centrode curves in the line space are investigated and some new results related to these integral invariants are obtained.

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Key words: Centrode curves, closed ruled surface and parallel p- äquidistante ruled surface.

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1. Introduction

Rather unexpectedly dual numbers which were developed in the mid-nineteenth century by Clifford have been applied to study the motion of a line space; they seem even to be the most appropriate apparatus for this end. It was first done by STUDY [14] and since his time dual numbers have an established place in kinematics as tool with which to attack problems dealing with lines in space .There exist a vast literature on subject (for example: BLASCHKE[1], BOTTEMA and ROTH[2],DIMENTBERG[3], HACISALİHOĞLU[6], ROTH and YANG[13], ŞENYURT[15], TÖLKE[16], YANG[18], YUAN[21], VELDKAMP[17]). The fundamental idea of Study's works is to replace points by lines as fundamental concepts of geometry. Points are then defined by the totality of straight lines passing through them. Oriented lines in three-dimensional space IR^3 may be represented by unit vectors with three components over the ring dual numbers. A differentiable curve on the dual unit sphere depending on the real parameter corresponds a ruled surface in IR^3. This correspondence is one-to-one and allows the geometry of ruled surface to be represented by the geometry of dual spherical curves on a dual unit sphere. Dual spherical geometry is closely analogues to real spherical geometry. Thus, the properties of elementary real spherical geometry can be carried over by analogy into the geometry of lines in IR^3 .It is known that a closed ruled surface has two integral invariants called pitch and the angle of pitch .The relationship between the pitch and the angle of pitch of a closed ruled surface is given by GÜRSOY [4-5], HOSCHEK[9] and MULLER[12]. Recently, several authors have used these invariants in their investigations concerning the generalization of some theorem of the plane kinematics to the spatial kinematics (for example: HACISALİHOĞLU [7-8], KÖSE [10], YAPAR [19-20]). On the other hand, parallel $p-$ äquidistante ruled surfaces in IR^3, which will be used in our study, have been studied by VALEONTIS [22] and generalized by MASAL [11].

The main intention of this paper, the relationship between the integral invariants of closed parallel $p-$äquidistante ruled surfaces generated by dual centrode curves in line space is investigated.

2. Preliminaries

The set of the dual numbers is

$ID = \{a + \varepsilon a^* : a, a^* \in IR, \varepsilon^2 = 0, \varepsilon \neq 0\}$.

This set is a commutative ring under addition and multiplication .This set can not be a field these operations, because $a + \varepsilon 0$ has no multiplication inverse ID.

The set of dual vectors is

$ID^3 = ID \times ID \times ID = \{X = x + \varepsilon x^* : x, x^* \in IR^3, \varepsilon \neq 0, \varepsilon^2 = 0\}$.

Scalar product on ID^3 is

$\langle , \rangle : ID^3 \times ID^3 \to ID$

$$(X,Y) \quad \to \langle X,Y \rangle = \langle x,y \rangle + \varepsilon \{\langle x^*, y \rangle + \langle x, y^* \rangle\}$$

where $x, y, x^*, y^* \in IR^3$.

Norm on ID^3 is

$\|,\| : ID^3 \to ID$

$$X \quad \to \|X\| = \|x\| + \varepsilon \frac{\langle x, x^* \rangle}{\|x\|} \quad , X \neq 0,$$

where the scalar product on the ring-hand side is the scalar products of IR^3.

Definition 2.1: The set K is called the unit dual sphere and defined by

(2.1) $$K = \{X = x + \varepsilon x^* : x, x^* \in IR^3, \|X\| = (1,0)\}.$$

Theorem 2.1: The oriented lines in IR^3 are in one-to-one correspondence with the points of the unit dual sphere in ID^3, [6].

Definition 2.2: Consider a non-singular linear transformation between two dual orthonormal systems rigidly linked to the unit dual spheres K (moving) and K' (fixed), respectively. In the case that corresponding matrix is an orthogonal matrix whose elements are differentiable dual functions of a real parameter t; the transformation is called one-parameter dual spherical motion and denoted by K/K', [6]. Moreover, if this matrix is a periodic function t, then the motion K/K' is a closed motion, otherwise it is called an open motion. During the

closed motion K/K', the orbit of the points of spheres K and K' is closed curves and thus the moving and fixed centrodes are closed curves. Then, the ruled surfaces corresponding to these curves are closed surfaces. The vector D is called dual Steiner vector defined by

(2.2) $$D = d + \varepsilon d^* = \oint_{(\alpha)} \psi$$

where ψ is the instantaneous Pfaffian vector of one-parameter dual spherical motion K/K'. For the real and dual parts of this vector, we can write

(2.3)
$$d = t \oint_{(\alpha)} \tau\, ds + b \oint_{(\alpha)} \kappa\, ds$$
$$d^* = t \oint_{(\alpha)} \tau^* ds + t^* \oint_{(\alpha)} \tau\, ds + b \oint_{(\alpha)} \kappa^* ds + b^* \oint_{(\alpha)} \kappa\, ds$$

where κ and τ are the curvature and torsion of the curve α on K. Let X be a dual point on K (moving).For the pitch, pitch angle and drall (distribution parameter) of the closed ruled surface generated by the dual point X in the lines space, we can write

(2.4)
$$L_X = \langle d, x^* \rangle + \langle d^*, x \rangle$$
$$\Lambda_X = -\langle D, X \rangle$$
$$P_X = \frac{\langle dx, dx^* \rangle}{\langle dx, dx \rangle} = \frac{d\varphi^*}{d\varphi}$$

where $d\varphi$ and $d\varphi^*$ are the angle and minimum distance between two neighbour directrix line corresponding to the unit dual vectors $X(s)$ and $X(s + \Delta s)$ in the ruled surfaces, respectively.Then, the dual spherical area bounded by the dual closed spherical curve $X = X(s)$ is calculated as; [6],

(2.5) $$F_X = 2\pi(1 - n) - \langle D, X \rangle.$$

For the real and dual parts of the vector $C = c + \varepsilon c^*$ on the motion K/K', we can write

$$c = (\sin\varphi)t + (\cos\varphi)b$$
$$c^* = (\sin\varphi)t^* + \varphi^*(\cos\varphi)t + (\cos\varphi)b^* - \varphi^*(\sin\varphi)b$$

where $\phi = \varphi + \varepsilon\varphi^*$. The pitch L_C, the pitch angle Λ_C and the drall P_C of the closed ruled surfaces corresponding to the fixed centrode curve generated by the vector C on K' in lines space can be given the following formulas, [15]:

(2.6)

$$L_C = (\sin\varphi)\oint_{(\alpha)} \tau^* ds + (\cos\varphi)\oint_{(\alpha)} \kappa^* ds + \varphi^*(\cos\varphi)\oint_{(\alpha)} \tau\, ds - \varphi^*(\sin\varphi)\oint_{(\alpha)} \kappa\, ds$$

$$\Lambda_C = -(\sin\varphi \oint_{(\alpha)} \tau\ ds + \cos\varphi \oint_{(\alpha)} \kappa\ ds) - \varepsilon\varphi^*(\cos\varphi \oint_{(\alpha)} \tau\, ds - \ \sin\varphi \oint_{(\alpha)} \kappa\, ds)$$

$$P_C = \frac{(\varphi^*)'}{\varphi'} = \frac{(\kappa^*\tau + \tau^*\kappa)(\kappa\kappa' - \tau\tau') + (\kappa^2 + \tau^2)[(\kappa^*)'\tau - \kappa(\tau^*)']}{(\kappa^2 + \tau^2)(\kappa\kappa' - \tau\tau')}$$

where $\overline{\kappa} = \kappa + \varepsilon\kappa^*$ and $\overline{\tau} = \tau + \varepsilon\tau^*$ are the dual curvature and torsion, respectively.

For the dual area of domain bounded by the dual spherical indicatrix of vector C, we can write, [20]:

(2.7) $F_C = 2\pi(1-n) - \langle D, C\rangle$.

Substituting by the value of $\langle D, C\rangle$ into the equation, we get

(2.8) $F_C = 2\pi(1-n) + \Lambda_C$.

Definition 2.3: Let S be a ruled surface in E^3. If $d = 0$, then S is called developable surface . Conversely, $d \neq 0$, then S is called skew ruled surface ,[22].If we take the striction curve of ruled surface S as the base curve, then the parametric equation of ruled surface S can be written

(2.9) $X(s,v) = \gamma(s) + vT(s)$.

Here, it suppose that the base curve r and striction line γ are differentiable class-C^2 and directrix vector T is differentiable class-C^3.

Definition 2.4: The planes corresponding to spaces $sp\{T,N\}, sp\{N,B\}$ and $sp\{T,B\}$ along the striction line of the ruled surface are called the asymptotic, polar and central planes, respectively, [22].

Definition 2.5 : Let S and S^* be two ruled surfaces in E^3.If the following two conditions are satisfied, the pair of ruled surfaces (S, S^*) is called parallel p - äquidistante ,[22]:

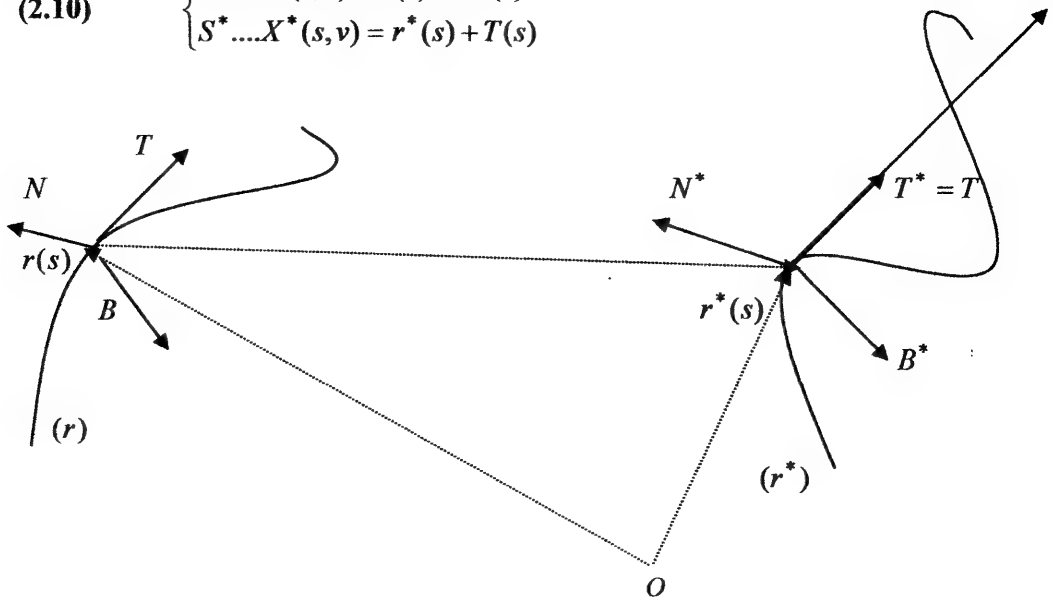
i) The generator vectors are parallel,

ii) The distance p between the polar planes at the suitable points of ruled surfaces S and S^* is constant.

Especially, if the polar planes are coincide, then the pair of ruled surface (S, S^*) is called parallel p -equivalent,[22].

For the parametrical representation of ruled surfaces S and S^*, we can write ,(fig.2.1):

(2.10)
$$\begin{cases} S.....X(s,v) = r(s) + vT(s) \\ S^*.....X^*(s,v) = r^*(s) + T(s) \end{cases}$$



(Figure 2.1)

If the base curves of the ruled surfaces S and S^* are taken as the striction lines γ and γ^*, we can write

(2.11)
$$\gamma^* = \gamma + zN + qB + pT$$

where $\{T, N, B\}$ is Frenet frames of curve r, [22] .

Theorem 2.1: Let S and S^* be parallel p -äquidistant ruled surfaces and the base curves of S and S^* be striction lines γ and γ^*, respectively. The striction line of S^* is given by

(2.12) $$\gamma^* = \gamma + zN + qB - (\frac{z' - q\tau}{\kappa})T$$

where the $z(s)$, $q(s)$ and $p(s)$ are differentiable functions class-C^2 and $r^* = \gamma + zN + qB + pT$,[22].

Theorem 2.2: Let S and S^* be parallel p -äquidistante ruled surfaces. The Frenet frames $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ at the points $r(s)$ and $r^*(s^*)$ of ruled surfaces S and S^* are congruent, [22].
Using the above theorem, we get

(2.13) $$\kappa = \kappa^* \frac{ds^*}{ds} \text{ and } \tau = \tau^* \frac{ds^*}{ds}.$$

Theorem 2.3: Let S and S^* be parallel p -äquidistant ruled surfaces. The following conditions are congruent, [22]:

1) The striction line γ of S is a helix curve.

2) The striction line γ^* of S^* is a helix curve.

3) The distance q between the asymptotic planes at suitable points of ruled surfaces S and S^* is constant.

3. The integral invariants of the closed ruled surfaces generated by a parallel p- äquidistante dual centrode curves in the lines space

Let S and $\overline{S}$ be the parallel p-äquidistante ruled surfaces in the dual spherical motion K/K' .For the dual Steiner vector $\overline{D}$ of $\overline{S}$, we can write

(3.1) $$\overline{D} = \overline{d} + \varepsilon \overline{d}^* = \oint_{\alpha} \overline{\Psi}$$

where $\overline{\Psi}$ is an instantaneous Pfaffian vector of $\overline{S}$. Because of the equations (2.4),(2.5),(2.6) and (2.7) for the integral invariants of the closed ruled surface generated by a point $\overline{X}$ taking on the moving dual sphere K in lines space we can write

(3.2) $$P_{\overline{X}} = \frac{\left\langle d\overline{x}, d\overline{x}^* \right\rangle}{\left\langle d\overline{x}, d\overline{x} \right\rangle} = \frac{d\overline{\varphi}^*}{d\varphi}$$

$$F_{\overline{X}} = 2\pi(1-n) - \left\langle \overline{D}, \overline{X} \right\rangle$$

where $\Phi = \overline{\varphi} + \varepsilon\overline{\varphi}^*$. Because of the equation (2.8), the real and dual parts of $\overline{C} = \overline{c} + \overline{c}^*$ are

$$\overline{c} = (\sin\overline{\varphi})\overline{t} + (\cos\overline{\varphi})\overline{b}$$

(3.3)

$$\overline{c}^* = (\sin\overline{\varphi})\overline{t}^* + \overline{\varphi}^*(\cos\varphi)\overline{t} + (\cos\overline{\varphi})\overline{b}^* - \overline{\varphi}^*(\sin\overline{\varphi})\overline{b}.$$

Because of the equations (2.9) and (2.10) the pitch length and pitch angle of the ruled surface corresponding to the fixed centrode curve generated by $\overline{C}$ on the sphere K' in the line space are

(3.4)
$$L_{\overline{C}} = \sin\varphi \oint_{\alpha} \overline{\tau}^* d\overline{s} + \cos\overline{\varphi} \oint_{\alpha} \overline{\kappa}^* d\overline{s} + \overline{\varphi}^* (\cos\overline{\varphi} \oint_{\alpha} \overline{\tau} d\overline{s} - \overline{\varphi}^* \sin\overline{\varphi} \oint_{\alpha} \overline{\kappa} d\overline{s})$$

and

(3.5)

$$\Lambda_{\overline{C}} = -(\sin\overline{\varphi} \oint_{\alpha} \overline{\tau} d\overline{s} + \cos\overline{\varphi} \oint_{\alpha} \overline{\kappa} d\overline{s}) - \varepsilon\overline{\varphi}^* (\cos\overline{\varphi} \oint_{\alpha} \overline{\tau} d\overline{s} - \sin\overline{\varphi} \oint_{\alpha} \overline{\kappa} d\overline{s})$$

where $\overline{\tau} = \overline{\tau} + \varepsilon\overline{\tau}^*$ and $\overline{\kappa} = \overline{\kappa} + \varepsilon\overline{\kappa}^*$ are the dual torsion and dual curvature of $\overline{\alpha}$, respectively.

Because of the equations (2.1) and (2.2), the drall $P_{\overline{C}}$ and the area $F_{\overline{C}}$ of ruled surface generated by the vector $\overline{C}$ are

(3.6)
$$P_{\overline{C}} = \frac{(\overline{\varphi}^*)'}{(\overline{\varphi})'}$$

and

(3.7)
$$F_{\overline{C}} = 2\pi(1-n) - \langle \overline{D}, \overline{C} \rangle .$$

Substituting by the equation (3.5) into the equation (3.7), we get

$$F_{\overline{C}} = 2\pi(1-n) - (\sin\overline{\varphi} \oint_{(\alpha)} \overline{\tau} d\overline{s} + \cos\overline{\varphi} \oint_{(\alpha)} \overline{\kappa} d\overline{s}) -$$

$$\varepsilon\overline{\varphi}^* (\cos\overline{\varphi} \oint_{(\alpha)} \overline{\tau} d\overline{s} + \sin\overline{\varphi} \oint_{(\alpha)} \overline{\kappa} d\overline{s})$$

or

(3.8) $$F_{\overline{C}} = 2\pi(1-n) - \Lambda_{\overline{C}}.$$

Since S and $\overline{S}$ are parallel p- äquidistante ruled surfaces we can write

(3.9) $$\begin{aligned} \sin\overline{\phi} &= \sin\phi \\ \cos\overline{\phi} &= \cos\phi \end{aligned}$$

or

(3.10) $$\overline{\phi} = \phi$$

where ϕ and $\overline{\phi}$ are dual angles.

For the dual and real parts of ϕ and $\overline{\phi}$, we can write as

(3.11) $$\begin{aligned} \overline{\varphi} &= \varphi \\ \overline{\varphi}^* &= \varphi^*. \end{aligned}$$

Substituting by the values of $\sin\overline{\varphi}, \cos\overline{\varphi}, \overline{\tau}^*, \overline{\kappa}^*$ and $\overline{\varphi}^*$ into the equation (3.4),we get

$$L_{\overline{C}} = (\sin\varphi)\oint_{\overline{\alpha}} \tau^* ds + (\cos\varphi)\oint_{\overline{\alpha}} \kappa^* ds + \varphi^*[\cos\varphi \oint_{\overline{\alpha}} \tau\, ds - \sin\varphi \oint_{\overline{\alpha}} \kappa\, ds]$$

Set the value of $\overline{\alpha}$ in the above equation, we obtain

$$L_{\overline{C}} = \sin\varphi \oint_{\gamma+ZN+QB+PT} \tau^* ds + \cos\varphi \oint_{\gamma+ZN+QB+PT} \kappa^* ds +$$

$$\varphi^*[\cos\varphi \oint_{\gamma+ZN+QB+PT} \tau\, ds - \sin\varphi \oint_{\gamma+ZN+QB+PT} \kappa\, ds$$

or

$$L_{\overline{C}} = (\sin\varphi)\oint_{\gamma} \tau^* ds + (\cos\varphi)\oint_{\gamma} \kappa^* ds + \varphi^*[\cos\varphi \oint_{\gamma} \tau\, ds - \sin\varphi \oint_{\gamma} \kappa\, ds] +$$

$$\{\sin\varphi \oint_{ZN+QB+PT} \tau^* ds + \cos\varphi \oint_{ZN+QB+PT} \kappa^* ds +$$

$$\varphi^*[\cos\varphi \oint_{ZN+QB+PT} \tau\, ds - \sin\varphi \oint_{ZN+QB+PT} \kappa\, ds]\}.$$

If we take the striction curve as the base curve of S , we get

(3.12) $$L_{\overline{C}} = L_C + e_1$$

where

$$e_1 = \sin\varphi \oint_{ZN+QB+PT} \tau^* ds + \cos\varphi \oint_{ZN+QB+PT} \kappa^* ds +$$
$$\varphi^* [\cos\varphi \oint_{ZN+QB+PT} \tau\, ds - \sin\varphi \oint_{ZN+QB+PT} \kappa\, ds].$$

Substituting by the values of $\sin\bar{\varphi}, \cos\bar{\varphi}^*, \bar{\tau}^*, \bar{\kappa}^*$ and $\bar{\varphi}^*$ into the equation (3.5), we get

$$\Lambda_{\bar{C}} = -(\sin\varphi \oint_{\bar{\alpha}} \bar{\tau} ds + \cos\varphi \oint_{\bar{\alpha}} \bar{\kappa} ds) -$$
$$\varepsilon\varphi^* (\cos\varphi \oint_{\bar{\alpha}} \tau ds - \sin\varphi \oint_{\bar{\alpha}} \kappa ds).$$

If we write the value of $\bar{\alpha}$ in the above equation, we obtain

$$\Lambda_{\bar{C}} = -(\sin\varphi \oint_{\gamma+ZN+QB+PT} \bar{\tau}\, ds + \cos\varphi \oint_{\gamma+ZN+QB+PT} \bar{\kappa}\, ds) -$$
$$\varepsilon\varphi^* (\cos\varphi \oint_{\gamma+ZN+QB+PT} \tau ds - \sin\varphi \oint_{\gamma+ZN+QB+PT} \kappa\, ds)$$

or

$$\Lambda_{\bar{C}} = -(\sin\varphi \oint_{\gamma} \bar{\tau} ds + \cos\varphi \oint_{\gamma} \bar{\kappa} ds) - \varepsilon\varphi^* (\cos\varphi \oint_{\gamma} \tau ds - \sin\varphi \oint_{\gamma} \kappa ds)$$
$$-\left\{(\sin\varphi \oint_{ZN+QB+PT} \bar{\tau}\, ds + \cos\varphi \oint_{ZN+QB+PT} \bar{\kappa}\, ds) + \right.$$

If we take the striction line as the base curve of S, we get

(3.13) $$\Lambda_{\bar{C}} = \Lambda_C + e_2$$

Where

$$e_2 = -\left\{(\sin\varphi \oint_{ZN+QB+PT} \bar{\tau}\, ds + \cos\varphi \oint_{ZN+QB+PT} \bar{\kappa}\, ds) + \right.$$
$$\left. \varepsilon\varphi^* (\cos\varphi \oint_{ZN+QB+PT} \tau ds - \sin\varphi \oint_{ZN+QB+PT} \kappa\, ds) \right\}.$$

Set the values of $\overline{\varphi}$ and $\overline{\varphi}^*$ in the equation (3.6), we obtain

(3.14) $P_{\overline{C}} = P_C$.

Substituting by the equation (3.13) into the equation (3.7), we get

$$F_{\overline{C}} = 2\pi(1-n) + \Lambda_C + e_2$$

and because of the equation (3.8)

(3.15) $F_{\overline{C}} = F_C + e_2$.

Thus, we can be given the following theorems:

Theorem 3.1: Let S and $\overline{S}$ be parallel p-äquidistante ruled surfaces. For the pitches $L_{\overline{C}}$ and L_C of the ruled surfaces generated by the curve $\overline{C}$ and C, we have

$$L_{\overline{C}} = L_C + e_1$$

Where

$$e_1 = \sin\varphi \oint_{ZN+QB+PT} \tau^* ds + \cos\varphi \oint_{ZN+QB+PT} \kappa^* ds + \varphi^* [\cos\varphi \oint_{ZN+QB+PT} \tau\, ds - \sin\varphi \oint_{ZN+QB+PT} \kappa\, ds].$$

Theorem 3.2: Let S and $\overline{S}$ be parallel p-äquidistante ruled surfaces. For the pitch angles $\Lambda_{\overline{C}}$ and Λ_C of the ruled surfaces generated by the curve C and $\overline{C}$, we have

$$\Lambda_{\overline{C}} = \Lambda_C + e_2$$

where

$$e_2 = -\left\{(\sin\varphi \oint_{ZN+QB+PT} \overline{\tau}\ ds + \cos\varphi \oint_{ZN+QB+PT} \overline{\kappa}\ ds) + \varepsilon\varphi^* (\cos\varphi \oint_{ZN+QB+PT} \tau ds - \sin\varphi \oint_{ZN+QB+PT} \kappa\ ds)\right\}.$$

Theorem 3.3: Let S and $\overline{S}$ be parallel p-äquidistante ruled surfaces. For the dralls (distribution parameters) $L_{\overline{C}}$ and L_C of the ruled surfaces generated by the curve $\overline{C}$ and C, we have

$$P_{\overline{C}} = P_C$$

Theorem 3.4: Let S and $\overline{S}$ be parallel p-äquidistante ruled surfaces. For the dual areas $F_{\overline{C}}$ and F_C of the ruled surfaces generated by the curve $\overline{C}$ and C, we have

$F_{\overline{C}} = F_C + e_2$.

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